ON FLAGGED FRAMED DEFORMATION PROBLEMS OF LOCAL CRYSTALLINE GALOIS REPRESENTATIONS

Kalloniatis, Tristan

Awarding institution:
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Tristan Kalloniatis

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The undersigned hereby certify that they have read and recommend to the Faculty of Natural and Mathematical Sciences for acceptance a thesis entitled "**On flagged framed deformation problems of local crystalline Galois representations**" by **Tristan Kalloniatis** in partial fulfillment of the requirements for the degree of **Doctor of Philosophy**.

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Research Supervisor: ______________________
Fred Diamond

Examining Committee: ______________________
Toby Gee

____
Jayanta Manoharmayum

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Author: **Tristan Kalloniatis**

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Signature of Author

To my parents for their continuing support over the years

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Abstract

In this thesis, we prove that irreducible residual Fontaine-Laffaille representations of the absolute Galois group of an unramified extension of $\mathbb{Q}_p$ have smooth representable crystalline framed deformation problems, provided that the Hodge-Tate weights lie in the Fontaine-Laffaille range. We then extend this result to the flagged lifting problem associated to any Fontaine-Laffaille upper triangular representation whose flag is of maximal length. We calculate the relative dimension of these various crystalline lifting functors in terms of the underlying Hodge-Tate weight structures, and also apply these results to give an alternative proof of the fact that every such residual representation admits a so-called "universally twistable lift". Finally we give some brief indications as to the various directions in which these results might be generalised.

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Introduction

The study of deformation problems of Galois representations has long been an important topic in algebraic number theory, both as a means to elucidate the structure of the associated Galois group, and to provide answers to more subtle arithmetic questions that can be tied to the existence of Galois representations with certain geometric properties. Chief amongst these is the application to modularity lifting theorems and the celebrated proof of Fermat's Last Theorem in [35]. Good expositions of these applications can be found in [29] and [32].

Fix a (rational) prime p and a finite field k of characteristic p. Letting $W(k)$ denote the Witt vectors of k, assume A is a local artinian $W(k)$-algebra with maximal ideal $\mathfrak{m}_A$ and residue field $k = A/\mathfrak{m}_A$. If G_K is the absolute Galois group of some field K and n is a positive integer then we consider (continuous) representations $\rho : G_K \longrightarrow GL_n(A)$ (with respect to the profinite topology on G_K and $\mathfrak{m}_A$-adic topology on $GL_n(A)$) with a specified reduction $\overline{\rho} : G_K \longrightarrow GL_n(k)$. In favourable circumstances, the collection of such lifts will be represented by a space $\mathrm{Spec}(R_{univ})$, with an associated universal lift $\rho : G_K \longrightarrow GL_n(R_{univ})$ of $\overline{\rho}$. An analogous statement holds when we consider lifts modulo strict equivalence (deformations).

Imposing conditions on the lifts (or deformations) allowed will cut down the space of allowed representations. If the conditions are chosen appropriately, we preserve the property of representability and can hope for some understanding of the representing spaces. Of particular interest when K is a global field are local conditions; for any completion $\hat{K}$ of K, we fix an embedding $\overline{K} \hookrightarrow \overline{\hat{K}}$ of algebraic closures and impose deformation conditions on the resulting restriction of ρ to $G_{\hat{K}}$.

For example, a 2-dimensional representation $\rho : G_{\mathbb{Q}} \longrightarrow GL_2(A)$ is said to be modular if it "arises from a modular form" (in the sense of Serre[1]). If the associated residual representation $\overline{\rho}$ satisfies certain technical conditions conditions, one imposes a deformation condition X which cuts out a subset $\mathrm{Spec}(R_X)$ of $\mathrm{Spec}(R_{univ})$. If additionally $\overline{\rho}$ is modular, one imposes a "modular" deformation condition which turns out to be represented by an appropriate Hecke algebra $\mathbb{T}_X$. To show that all deformations satisfying X are modular, it then suffices to show that $R_X = \mathbb{T}_X$; this is done in [35].

For this thesis, we instead focus on the situation where K is a finite extension of $\mathbb{Q}_p$ and discuss the notion of crystalline representations. Examples of crystalline representations in characteristic zero are those arising from the étale cohomology of proper varieties over K with a smooth model over the ring of integers $\mathcal{O}_K$ due to the crystalline comparison theorem in cohomology (see [33]; the reader is advised to also consult [16]), and are parametrised in general via the theory of filtered ϕ-modules (see [11]). To define crystalline representations over local artinian $W(k)$-algebras with residue field k as above, and thus approach the theory of crystalline framed deformation problems, we may in sufficiently simple situations[2] use the theory of Fontaine-Laffaille modules discussed in [19]. These objects are sufficiently

[1]For more details on this construction, the reader is advised to consult [31] and [13].

[2]Namely when K is unramified over $\mathbb{Q}_p$ and all labelled Hodge-Tate weights differ by at most $p-2$.

explicit that we can actually parametrise all lifts in terms of filtration preserving endomorphisms of the underlying filtered module. Adapting an argument from [28], we may then prove the following theorem.

Theorem A. *The crystalline framed deformation functor associated to an absolutely irreducible rank n Fontaine-Laffaille residual representation $\overline{\rho}$ of the Galois group of a finite unramified extension K of $\mathbb{Q}_p$ with labelled Hodge-Tate weights differing by at most $p-2$ is smoothly representable, and of relative dimension $n^2([K:\mathbb{Q}_p]+1)-d_{\overline{\rho}}$, where $d_{\overline{\rho}}$ is an explicit quantity depending only on the labelled Hodge-Tate weight structure of $\overline{\rho}$.*

Having done this, we then consider residual Fontaine-Laffaille representations as above together with a flag of maximal length. These take the form

$$\overline{\rho} = \begin{pmatrix} \overline{\rho_1} & * & \dots & * \\ 0 & \overline{\rho_2} & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \overline{\rho_r} \end{pmatrix}$$

for $\overline{\rho_i}$ irreducible Fontaine-Laffaille representations of rank n_i $(i = 1, 2, \dots, r)$. A framed deformation problem is specified by lifting the maximal flag, giving lifts ρ of this same form. By relating crystalline extensions of representations to filtration preserving homomorphisms between the corresponding Fontaine-Laffaille modules, a simple induction argument allows us to prove the following theorem.

Theorem B. *The flagged crystalline framed deformation functor associated to $\overline{\rho}$ as above with labelled Hodge-Tate weights differing by at most $p-2$ is smoothly representable of relative dimension*

$$\sum_{i=1}^{r}(\sum_{j<i} n_j)n_i([K:\mathbb{Q}_p]+1) - d_{\overline{\rho_{<i}},\overline{\rho_i}}$$

where $d_{\overline{\rho_{<i}},\overline{\rho_i}}$ *is an explicit quantity depending only on the labelled Hodge-Tate weight structure of the representations* $\overline{\rho_i}$ *(*$i = 1, 2, \ldots, r$*).*

Finally, the above theorems give a simple deduction of one of the main theorems of [20] on "universally twistable lifts", a notion which will be defined in Definition 3.3.1.

Theorem C. *Suppose* K *is unramified over* $\mathbb{Q}_p$*, and let* $\overline{\rho}$ *be a rank* n *Fontaine-Laffaille residual representation of* G_K *together with a maximal flag. Then* $\overline{\rho}$ *admits a universally twistable lift.*

All of these results rely heavily on the theory of Fontaine-Laffaille modules and so any significant generalisations would likely require additional theoretical input, for example the theory of Wach modules discussed in [34].

0.1 Outline of this thesis

Chapter 1 is preliminary. In it, we give a brief review of the basic concepts of representable functors and deformation theory that will be needed throughout the rest of the thesis, as well as a primer on the structure of local Galois groups and their irreducible mod p representations.

Chapter 2 is devoted to introducing the concept of crystalline/Fontaine-Laffaille representations, particularly in characteristic p, and the associated semilinear algebraic objects that we use to study them via integral p-adic Hodge theory. This culminates in some results on the structure of extensions of such objects.

Chapter 3 proves all of the advertised results, in particular the smooth representability of the crystalline deformation functor associated to a block upper triangular residual Fontaine-Laffaille representation of an unramified Galois group (Theorem B, proved as Corollary 3.1.6; the special case of a length 1 flag, Theorem A, is proved as Theorem 3.1.1). We calculate the relative dimension of this functor explicitly in terms of the labelled Hodge-Tate weight structure of the associated Fontaine-Laffaille module, and give some simple bounds in section 3.4. We apply these results to show that all such representations admit a "universally twistable lift" (Theorem C, proved as Theorem 3.3.6). We end with some indications as to how the results on smooth representability might be extended, especially when the condition on the range of allowable Hodge-Tate weights is relaxed.

0.2 Notational conventions

Throughout, p denotes a fixed prime number. K is a field, almost always a finite unramified extension of $\mathbb{Q}_p$, in which case we let Frob (or sometimes ϕ) denote arithmetic Frobenius on K. L is a finite extension of $\mathbb{Q}_p$ containing the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, with residue field k_L. The p-adic valuation v_p will be normalised so that $v_p(p) = 1$, and we write $\mathcal{O}_K$ and $\mathcal{O}_L$ for the rings of integers of K and L, respectively. We let $\mathbb{C}_K$ denote the completion of the algebraic closure of K, with ring of integers $\mathcal{O}_{\mathbb{C}_K}$. We write G_K for the absolute Galois group of K, χ_p for the p-adic cyclotomic character of G_K, and adopt the sign convention that χ_p has all labelled Hodge-Tate weights equal to $+1$.

For a ring R and a finite module M over R, we write $\mathrm{lg}_R(M)$ for the length of M as an R-module.

We write $\mathcal{C}_L$ for the category of complete local artinian $\mathcal{O}_L$-algebras with residue field k_L, and $\hat{\mathcal{C}_L}$ for the category of complete local noetherian $\mathcal{O}_L$-algebras with residue field k_L; in both cases, morphisms are local $\mathcal{O}_L$-algebra homomorphisms reducing to the identity on residue fields. For $A \in \mathcal{C}_L$ (or $A \in \hat{\mathcal{C}_L}$), we write $\mathfrak{m}_A$ for the maximal ideal of A.

0.2.1 Glossary of common categories used in this thesis

Notation	Definition	First used
$\mathcal{C}_L$	Local artinian $\mathcal{O}_L$-algebras with residue field k_L	7
Mod_R	Free finite rank modules over R	22
$\mathrm{Fil}(K)$	Finite dimensional vector spaces over K with a decreasing exhaustive separated filtration	26
$\mathrm{Rep}_L^{cris}(G_K)$	Crystalline representations of G_K with coefficients in L	27
$MF_K^{\phi} \bigotimes_{\mathbb{Q}_p} L$	Filtered ϕ-modules over K with coefficients in L	28
$MF_{tor,\mathcal{O}_K}^{f,r} \bigotimes_{\mathbb{Z}_p} A$	Fontaine-Laffaille modules over A with weights at most r	38
$\mathrm{Rep}_{\mathbb{Z}_p}^{f}(G_K)$	Finite length $\mathbb{Z}_p$-representations of G_K	40
$\mathrm{Rep}_A^{cris,\leq r}(G_K)$	Crystalline/Fontaine-Laffaille representations of G_K with coefficients in A and Hodge-Tate weights between 0 and r	41
Fil, R	Free finite rank modules over K with a decreasing exhaustive separated filtration	48
$M_{\mathbf{A}_{K,L}}^{\phi,\Gamma,\acute{e}t}$	Étale (ϕ,Γ)-modules over $\mathbf{A}_{K,L}$	68
$M_{\mathbf{B}_{K,L}}^{\phi,\Gamma,\acute{e}t}$	Étale (ϕ,Γ)-modules over $\mathbf{B}_{K,L}$	68

Chapter 1

Representability of Galois deformation problems and local Galois groups

In this preliminary chapter, we review the relevant material on deformation problems, representablity, and deformation conditions, as well as some basic results on the structure and representation theory of local Galois groups, that will be needed in later chapters.

1.1 Deformation theory

In this section we will review the theory of deformation problems, especially as applicable to Galois representations. For a more detailed explanation of the theory, the reader is advised to consult [23] and [25].

Throughout this section, we will fix the following notation. Let G be a profinite group, p a prime, L a finite extension of $\mathbb{Q}_p$ with ring of integers $\mathcal{O}_L$ and residue field k_L. Let $\mathcal{C}_L$ be the category of local artinian $\mathcal{O}_L$-algebras with residue field k_L; morphisms in $\mathcal{C}_L$ are local $\mathcal{O}_L$-algebra homomorphisms which reduce to the identity on residue fields. If $A \in \mathcal{C}_L$, we write $\mathfrak{m}_A$ for the maximal ideal of A. Occasionally we will make use of the related category $\hat{\mathcal{C}_L}$ whose objects are complete local noetherian $\mathcal{O}_L$-algebras with residue field k_L and morphisms as in $\mathcal{C}_L$. We will also fix a positive integer n and a (continuous) residual representation $\overline{\rho} : G \longrightarrow GL_n(k_L)$.

1.1.1 General theory and representability

Definition 1.1.1. G satisfies the *p-finiteness condition* if for every open subgroup $H \subseteq G$, the set $\mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})$ of continuous homomorphisms is finite (where $\mathbb{Z}/p\mathbb{Z}$ is given the discrete topology).

From now on we will assume that the group G satisfies the p-finiteness condition. In practice throughout this thesis, G will be the absolute Galois group $G_K = \mathrm{Gal}(\overline{K}/K)$ of a p-adic field K; in this case, finiteness of $\mathrm{Hom}(H, \mathbb{Z}/p\mathbb{Z})$ follows from local class field theory for the field $\overline{K}^H$ (see Proposition 1.2.1).

Definition 1.1.2.

1. The *framed deformation functor* $\mathcal{D}^{\square}_{\overline{\rho}} : \mathcal{C}_L \longrightarrow \mathrm{Set}$ associated to $\overline{\rho}$ is the functor which associates to $A \in \mathcal{C}_L$ the set of (continuous) lifts $\rho : G \longrightarrow GL_n(A)$ of $\overline{\rho}$ to A, and sends morphisms to the natural map.

2. The *deformation functor* $\mathcal{D}_{\overline{\rho}} : \mathcal{C}_L \longrightarrow \mathrm{Set}$ associated to $\overline{\rho}$ is the functor which associates to $A \in \mathcal{C}_L$ the set of strict equivalence classes of (continuous) lifts $\rho : G \longrightarrow GL_n(A)$ of $\overline{\rho}$ to A. Here, representations $\rho_1, \rho_2 : G_K \longrightarrow GL_n(A)$ are *strictly equivalent* if there is an element $\gamma \in \mathrm{Ker}(GL_n(A) \longrightarrow GL_n(k_L))$ such that ρ_1 and ρ_2 are conjugate by γ.

We will occasionally need to consider the value of a functor $\mathcal{F} : \mathcal{C}_L \longrightarrow \mathrm{Set}$ on an element of $\hat{\mathcal{C}_L}$. To this end, we make the following definition.

Definition 1.1.3. A covariant functor $\mathcal{F} : \hat{\mathcal{C}_L} \longrightarrow \mathrm{Set}$ is *continuous* if for all $A \in \hat{\mathcal{C}_L}$, the natural map

$$\mathcal{F}(A) \longrightarrow \lim_{\leftarrow} \mathcal{F}(A/\mathfrak{m}_A^r)$$

is a bijection.

Example 1.1.4. For any residual representation $\overline{\rho} : G \longrightarrow GL_n(k_L)$, the functors $\mathcal{D}_{\overline{\rho}}^{\square}$ and $\mathcal{D}_{\overline{\rho}}$ (extended in the natural way to $\hat{\mathcal{C}_L}$) are continuous. Indeed this is clear for $\mathcal{D}_{\overline{\rho}}^{\square}$, while for $\mathcal{D}_{\overline{\rho}}$ we argue as follows.

1. $\mathcal{D}_{\overline{\rho}}(A) \longrightarrow \lim\limits_{\leftarrow} \mathcal{D}_{\overline{\rho}}(A/\mathfrak{m}_A^r)$ is injective: given lifts ρ, ρ' of $\overline{\rho}$ to A and elements $\gamma_r' \in \mathrm{Ker}(GL_n(A/\mathfrak{m}_A^r) \longrightarrow GL_n(k_L))$ making $\rho \pmod{\mathfrak{m}_A^r}$ conjugate to $\rho' \pmod{\mathfrak{m}_A^r}$, lift $(\gamma_r')_{r=1}^{\infty}$ to a sequence $(\gamma_r)_{r=1}^{\infty} \in \mathrm{Ker}(GL_n(A) \longrightarrow GL_n(k_L))$. Then $(\gamma_r)_{r=1}^{\infty}$ has an accumulation point $\gamma \in \mathrm{Ker}(GL_n(A) \longrightarrow GL_n(k_L))$ making ρ conjugate to ρ', by compactness.

2. $\mathcal{D}_{\overline{\rho}}(A) \longrightarrow \lim\limits_{\leftarrow} \mathcal{D}_{\overline{\rho}}(A/\mathfrak{m}_A^r)$ is surjective: given a sequence $\rho_r : G \longrightarrow GL_n(A/\mathfrak{m}_A^r)$ of lifts, and elements $\gamma_r'' \in \mathrm{Ker}(GL_n(A/\mathfrak{m}_A^r) \longrightarrow GL_n(k_L))$ such that $\rho_{r+1} \equiv \gamma_r''\rho_r(\gamma_r'')^{-1} \pmod{\mathfrak{m}_A^r}$, lift the sequence $(\gamma_r'')_{r=1}^{\infty}$ to a sequence $(\gamma_r')_{r=1}^{\infty} \in \mathrm{Ker}(GL_n(A) \longrightarrow GL_n(k_L))$ and put $\gamma_r = \prod_{j<r} \gamma_r'$. Then $\gamma_r\rho_r\gamma_r^{-1}$ is a compatible sequence of lifts to $A/\mathfrak{m}_A^r$, so defines a lift ρ to A whose strict equivalence class maps to $([\rho_r])_{r=1}^{\infty}$.

Definition 1.1.5. A covariant functor $\mathcal{F} : \mathcal{C}_L \longrightarrow \mathrm{Set}$ is *(pro-)representable* if there is an object of $\hat{\mathcal{C}}_L$, denoted $R(\mathcal{F})$, such that for all $A \in \mathcal{C}_L$, we have $\mathcal{F}(A) = \mathrm{Hom}_{\hat{\mathcal{C}}_L}(R(\mathcal{F}), A)$.

We then have the following fundamental result.

Theorem 1.1.6. *Assume that G satisfies the p-finiteness condition.*

1. *$\mathcal{D}^{\square}_{\overline{\rho}}$ is (pro-)representable by an object $R^{\square}_{\overline{\rho}} \in \hat{\mathcal{C}}_L$.*

2. *Assume additionally that $\overline{\rho}$ is absolutely irreducible*[1]*. Then $\mathcal{D}_{\overline{\rho}}$ is (pro-)representable by an object $R_{\overline{\rho}} \in \hat{\mathcal{C}}_L$.*

Proof. See [3], Proposition 1.3.1. □

Remark 1.1.7. Since $\mathcal{D}^{\square}_{\overline{\rho}}$ (respectively, $\mathcal{D}_{\overline{\rho}}$) is continuous, it is useful to think of the lift (respectively, deformation) to $R^{\square}_{\overline{\rho}}$ (respectively, $R_{\overline{\rho}}$ when $\overline{\rho}$ is absolutely irreducible) corresponding to the identity morphism. This is referred to as the *universal lift* (respectively, *universal deformation*) of $\overline{\rho}$.

1.1.2 Tangent spaces, Schlessinger's criterion, and smoothness

In this section we briefly recall some of the theory necessary to talk about properties of (pro-)representable functors. Good references for an overview include [25], [26], and [3].

Definition 1.1.8. 1. $k_L[\epsilon]$, the set of *dual numbers*, denotes the ring obtained by adjoining an element ϵ to k_L and demanding $\epsilon^2 = 0$.

[1] *In fact, it suffices to assume that $\overline{\rho}$ is* Schur*; that is, that $End_{k_L[G_K]}(\overline{\rho}) = k_L$.*

2. Let $\mathcal{F} : \mathcal{C}_L \longrightarrow$ Set be a covariant functor such that $\mathcal{F}(k_L)$ consists of one point. The *tangent space* of $\mathcal{F}$, denoted $t_{\mathcal{F}}$, is the set $\mathcal{F}(k_L[\epsilon])$.

For the remainder of this section, we shall assume that all functors $\mathcal{F} : \mathcal{C}_L \longrightarrow$ Set are covariant and have $\mathcal{F}(k_L)$ consisting of a single point.

Proposition 1.1.9. *Suppose $\mathcal{F}$ as above is (pro-)representable by a ring $R(\mathcal{F})$. Then $t_{\mathcal{F}} \cong Hom_{k_L}(\frac{m_{R(\mathcal{F})}}{m^2_{R(\mathcal{F})}+pR(\mathcal{F})}, k_L)$, where Hom_{k_L} denotes k_L-linear maps.*

Proof. See [26], Proposition 15. □

In particular, $t_{\mathcal{F}}$ obtains the structure of a finite dimensional k_L-vector space.

Lemma 1.1.10. *Suppose $\mathcal{F}$ and $\mathcal{G}$ are functors as above, (pro-)representable by objects $R(\mathcal{F})$ and $R(\mathcal{G})$ respectively, and let $\mathcal{F} \longrightarrow \mathcal{G}$ be a natural transformation of functors. Then $t_{\mathcal{F}} \hookrightarrow t_{\mathcal{G}}$ if and only if the induced map $R(\mathcal{G}) \longrightarrow R(\mathcal{F})$ is surjective.*

Proof. By the above proposition, if $t_{\mathcal{F}} \hookrightarrow t_{\mathcal{G}}$ then $\frac{m_{R(\mathcal{G})}}{m^2_{R(\mathcal{G})}+pR(\mathcal{G})} \twoheadrightarrow \frac{m_{R(\mathcal{F})}}{m^2_{R(\mathcal{F})}+pR(\mathcal{F})}$ is surjective; it follows from [30] (Lemma 1.1) that $R(\mathcal{G}) \twoheadrightarrow R(\mathcal{F})$ is surjective. The converse is immediate. □

Corollary 1.1.11. *If a functor $\mathcal{F}$ as above is (pro-)representable by an object $R(\mathcal{F})$, and $dim_{k_L} t_{\mathcal{F}} = r$, then there is a surjection $\mathcal{O}_L[[X_1, X_2, \ldots, X_r]] \twoheadrightarrow R(\mathcal{F})$.*

Proof. Pick $x_1, x_2, \ldots, x_r \in m_{R(\mathcal{F})}$ whose images in $\frac{m_{R(\mathcal{F})}}{m^2_{R(\mathcal{F})}+pR(\mathcal{F})}$ are linearly independent and define a map $\mathcal{O}_L[[X_1, X_2, \ldots, X_r]] \longrightarrow R(\mathcal{F})$ by sending $X_i \mapsto x_i$ $(i = 1, 2, \ldots, r)$. The result then follows from Nakayama's lemma and the above lemma. □

We focus now on the case where $\mathcal{F} = \mathcal{D}^{\square}_{\overline{\rho}}$ is the framed deformation functor (or occasionally deformation functor $\mathcal{D}_{\overline{\rho}}$) of some residual representation $\overline{\rho}$. We then have the following fundamental interpretation of the tangent space.

Proposition 1.1.12. *$t_{\mathcal{D}^{\square}_{\overline{\rho}}}$ is naturally identified with the set of (continuous) cocycles $Z^1(G, End_{k_L}(\overline{\rho}))$, where G acts on $End_{k_L}(\overline{\rho})$ by conjugation. Similarly, we have an identification $t_{\mathcal{D}_{\overline{\rho}}} \cong H^1(G, End_{k_L}(\overline{\rho}))$.*

Proof. We sketch a proof. For more details, see [26], Proposition 21.1.

The exact sequence

$$1 \longrightarrow \mathrm{Ker}(GL_n(k_L[\epsilon]) \longrightarrow GL_n(k_L)) \longrightarrow GL_n(k_L[\epsilon]) \longrightarrow GL_n(k_L) \longrightarrow 1$$

is split by the natural inclusion $GL_n(k_L) \subseteq GL_n(k_L[\epsilon])$; moreover, we identify $\mathrm{Ker}(GL_n(k_L[\epsilon]) \longrightarrow GL_n(k_L))$ with $\mathrm{End}_{k_L}(\overline{\rho})$ by sending a matrix $I + \epsilon X \mapsto X$. In this way, $GL_n(k_L)$ acts on $\mathrm{End}_{k_L}(\overline{\rho})$ via conjugation in the presentation of the semidirect product $GL_n(k_L[\epsilon]) \cong End_{k_L}(\overline{\rho}) \rtimes GL_n(k_L)$.

Let ρ_0 denote the element of $t_{\mathcal{D}^{\square}_{\overline{\rho}}}$ obtained by composing $\overline{\rho}$ with the inclusion $GL_n(k_L) \subseteq GL_n(k_L[\epsilon])$; then a point $\rho \in t_{\mathcal{D}^{\square}_{\overline{\rho}}}$ gives rise to the difference cocycle $c_\rho : G \longrightarrow \mathrm{End}_{k_L}(\overline{\rho})$, where $c_\rho(g) = \rho(g)\rho_0(g)^{-1}$. One checks that $\rho \mapsto c_\rho$ is the required identification. For the deformation functor $\mathcal{D}_{\overline{\rho}}$, one must additionally check that strict equivalence of lifts corresponds precisely to cohomology of cocycles. $\square$

We also mention the following useful result, known as Schlessinger's criterion.

Proposition 1.1.13. *Let $\mathcal{F} : \mathcal{C}_L \longrightarrow Set$ be a covariant functor such that $\mathcal{F}(k_L)$ consists of one point. For any $A, B, C \in \mathcal{C}_L$ and maps $A \longrightarrow B$ and $C \longrightarrow B$, consider the corresponding map*

$$\mathcal{F}(A \times_B C) \longrightarrow \mathcal{F}(A) \times_{\mathcal{F}(B)} \mathcal{F}(C).$$

Then $\mathcal{F}$ is (pro-)representable by an object of $\hat{\mathcal{C}_L}$ if and only if the following four criteria are met:

1. *The above map is surjective whenever $A \longrightarrow B$ is a small extension in the sense of the definition below.*

2. *The above map is bijective for $B = k_L$, $C = k_L[\epsilon]$, A arbitrary.*

3. *The above map is bijective whenever $A = C$ and the maps $A \longrightarrow B$, $C \longrightarrow B$ are equal and small extensions in the sense of the definition below.*

4. $dim_{k_L}(t_{\mathcal{F}}) < \infty$.

Proof. See [30], Theorem 2.11. □

Definition 1.1.14. A map $A \longrightarrow B$ of objects of $\mathcal{C}_L$ is a *small extension* if it is surjective with kernel a nonzero principal ideal (t) of A such that $t \cdot \mathfrak{m}_A = 0$.

We finish this section with a brief discussion of the notions of (formal) smoothness and relative dimension which will feature prominently in chapter 3.

Definition 1.1.15. A morphism of functors $\mathcal{F} \longrightarrow \mathcal{G}$ is *smooth* if for all surjections $A \twoheadrightarrow B$ of objects of $\mathcal{C}_L$, the natural map

$$\mathcal{F}(A) \longrightarrow \mathcal{F}(B) \times_{\mathcal{G}(B)} \mathcal{G}(A)$$

is surjective.

Example 1.1.16. For any irreducible $\overline{\rho} : G \longrightarrow GL_n(k_L)$, the morphism $\mathcal{D}^{\square}_{\overline{\rho}} \longrightarrow \mathcal{D}_{\overline{\rho}}$ (obtained by sending a lift of $\overline{\rho}$ to its strict equivalence class) is smooth. Indeed, given $\theta : A \twoheadrightarrow B$ as in the above definition, a lift ρ to B and a deformation $[\rho']$ to A such that $[\theta\rho'] = [\rho]$, pick $\gamma' \in \mathrm{Ker}(GL_n(B) \longrightarrow GL_n(k_L))$ such that $\gamma'\theta\rho'(\gamma')^{-1} = \rho$, and lift it to $\gamma \in \mathrm{Ker}(GL_n(A) \longrightarrow GL_n(k_L))$. Then the lift $\gamma\rho'\gamma^{-1}$ to A lies above ρ and $[\rho']$.

Our principal interest in this definition is due to the following result.

Proposition 1.1.17. *A morphism $\mathcal{F} \longrightarrow \mathcal{G}$ of (pro-)representable functors is smooth if and only if $R(\mathcal{F})$ is a power series ring over $R(\mathcal{G})$.*

Proof. See [30], Proposition 2.5. □

In the situation of the above proposition, the *relative dimension* of the morphism $\mathcal{F} \longrightarrow \mathcal{G}$ is simply the number of variables required to write $R(\mathcal{F})$ as a power series ring over $R(\mathcal{G})$.

1.1.3 Deformation problems

Fix all assumptions and notation as in the previous section. We are interested in studying subfunctors of the framed deformation functor $\mathcal{D}^{\square}_{\overline{\rho}}$ with a particular property. This definition is based on Definition 1 of [28] and the discussion following it.

Definition 1.1.18. Let X be a property of finite $\mathcal{O}_L[G]$-modules that is closed under direct sums and subquotients. Then for $A \in \mathcal{C}_L$, we define $\mathcal{D}^{\square,X}_{\overline{\rho}}(A)$ as the set of $\rho \in \mathcal{D}^{\square}_{\overline{\rho}}(A)$ whose underlying finite $\mathcal{O}_L[G]$-module has property X. Such a property X is called a *deformation condition* (for $\overline{\rho}$).

Proposition 1.1.19. *Let X be a deformation condition for $\overline{\rho}$.*

1. $\mathcal{D}_{\overline{\rho}}^{\square,X}$ *is a subfunctor of* $\mathcal{D}_{\overline{\rho}}^{\square}$.

2. $\mathcal{D}_{\overline{\rho}}^{\square,X}$ *is (pro-)representable by an object* $R_{\overline{\rho}}^{\square,X} \in \hat{\mathcal{C}}_L$*, which is a quotient of* $R_{\overline{\rho}}^{\square}$*. That is,* $\mathcal{D}_{\overline{\rho}}^{\square,X} \subseteq \mathcal{D}_{\overline{\rho}}^{\square}$ *is "closed".*

Proof. The argument for the proof of both of these statements is essentially the content of section 1 of [28] (specifically Theorem 1.1, and Propositions 1.1 and 1.2), with trivial modifications made to deal with framed deformation functors as opposed to deformation functors. We give some details here for convenience.

1. Given $A, B \in \mathcal{C}_L$, a $\mathcal{C}_L$-morphism $\phi : A \longrightarrow B$, and an object $\rho \in \mathcal{D}_{\overline{\rho}}^{\square,X}(A)$, it suffices to show that $\phi\rho \in \mathcal{D}_{\overline{\rho}}^{\square,X}(B)$. Consider A^n and B^n as rings with G acting via ρ and $\phi\rho$ respectively, so $\phi^n : A^n \longrightarrow B^n$ is G-equivariant. By assumption, A^n has property X. Since $B \in \mathcal{C}_L$, B^n is finite and so finitely generated over A^n; in other words there exists m and a surjection $(A^n)^m \twoheadrightarrow B^n$. $(A^n)^m$ has property X since X is closed under direct sums, and so the same is true of B^n since X is closed under quotients.

2. We first show $\mathcal{D}_{\overline{\rho}}^{\square,X}$ is (pro-)representable by using Proposition 1.1.13. Given $A, B, C \in \mathcal{C}_L$ and maps $A \longrightarrow B$ and $C \longrightarrow B$, we check that if $A \longrightarrow B$ is a small extension then $\mathcal{D}_{\overline{\rho}}^{\square,X}(A \times_B C) \twoheadrightarrow \mathcal{D}_{\overline{\rho}}^{\square,X}(A) \times_{\mathcal{D}_{\overline{\rho}}^{\square,X}(B)} \mathcal{D}_{\overline{\rho}}^{\square,X}(C)$. Let $(\rho_A, \rho_C) \in \mathcal{D}_{\overline{\rho}}^{\square,X}(A) \times_{\mathcal{D}_{\overline{\rho}}^{\square,X}(B)} \mathcal{D}_{\overline{\rho}}^{\square,X}(C)$. Since $\mathcal{D}_{\overline{\rho}}^{\square}$ is (pro-)representable, there is some $\rho \in \mathcal{D}_{\overline{\rho}}^{\square}(A \times_B C)$ mapping to (ρ_A, ρ_C); it remains to show that ρ has property X. But since $A \times_B C \hookrightarrow A \times C$ is injective, it follows that the map $(A \times_B C)^n \hookrightarrow A^n \times C^n$ is an injective map of G-modules, with target

having property X since X is closed under direct sums; thus the source also has property X since X is closed under taking subobjects. In other words, ρ has property X.

We have just verified condition 1 of 1.1.13. The remaining conditions then follow from Proposition 1.1.19 together with the fact that $\mathcal{D}_{\overline{\rho}}^{\square}$ is (pro-)representable.

Finally, we must show that the map $\theta : R_{\overline{\rho}}^{\square} \longrightarrow R_{\overline{\rho}}^{\square,X}$ is surjective. This follows from part 1 together with an application of Lemma 1.1.10. $\square$

The key example of a Galois deformation problem that will be relevant to this thesis is the notion of crystalline deformations. This will be discussed in the following chapter, in particular Proposition 2.2.14.

1.2 Structure of local Galois groups

Finally in this chapter, we briefly recall some results on the structure of local Galois groups that will be needed later on. Throughout, we will fix notation as above, with the additional assumptions that K be a finite extension of $\mathbb{Q}_p$, $G = G_K$ be the absolute Galois group of K, and L be a p-adic field containing the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$. Fix a uniformiser ϖ of K, and let q denote the size of the residue field of K.

Let K^{nr} denote the maximal unramified extension of K and K^t the maximal tamely ramified extension of K inside $\overline{\mathbb{Q}_p}$. We denote the associated Galois groups as $I_K = \mathrm{Gal}(\overline{K}/K^{nr})$ and $P_K = \mathrm{Gal}(\overline{K}/K^t)$. The following facts are then standard:

1. $K^{nr} = \bigcup_{p\nmid n} K(\zeta_n)$ for ζ_n a primitive root of unity of order n (prime to p).

Moreover, $G_K/I_K = \hat{\mathbb{Z}}$, topologically generated by (arithmetic) Frobenius ϕ which acts on ζ_n as $\phi(\zeta_n) = \zeta_n^q$.

2. $I_K/P_K \cong \prod_l \mathbb{Z}_l$ where the product runs over primes $l \neq p$. The $\mathbb{Z}_l$-factor corresponds to the extension $\bigcup_{m=1}^{\infty} K^{nr}(\sqrt[l^m]{\varpi})$ of K^{nr}, and the Galois group is topologically generated by an element σ_l which acts on a compatible choice $\sqrt[l^m]{\varpi}$ of roots of ϖ as $\sigma_l(\sqrt[l^m]{\varpi}) = \zeta_{l^m}\sqrt[l^m]{\varpi}$.

3. We can lift ϕ to G_K/P_K in such a way that this group is topologically generated by ϕ and the collection σ_l for $l \neq p$, subject to the relations that the σ_l all commute with each other, and that $\phi\sigma_l\phi^{-1} = \sigma_l^q$.

4. P_K is a pro-p group, and a Sylow p-subgroup of I_K.

We also have the following fundamental result of local class field theory; here we choose the normalisation so that the Artin map sends a uniformiser of K to a lift of arithmetic Frobenius.

Proposition 1.2.1. *Let G_K^{ab} denote the abelianisation of G_K. Then there is an injection $r_K : K^\times \hookrightarrow G_K^{ab}$, the* Artin map*, which has dense image and fits into the following commutative diagram*

$$\begin{array}{ccccccccc}
1 & \longrightarrow & \mathcal{O}_K^\times & \longrightarrow & K^\times & \longrightarrow & \mathbb{Z} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
1 & \longrightarrow & Gal(K^{ab}/K^{nr}) & \longrightarrow & G_K^{ab} & \longrightarrow & G_K/I_K & \longrightarrow & 0
\end{array}$$

where the leftmost vertical arrow is an isomorphism, the middle arrow is the Artin map, and the rightmost arrow is the natural inclusion $\mathbb{Z} \hookrightarrow \hat{\mathbb{Z}}$.

Now for a natural number n, let K_n denote the unique unramified extension of K of degree n.

Definition 1.2.2. Pick $\pi_n \in \overline{K}$ such that $\pi_n^{p^n-1} = \varpi$. We then define a character $\omega_n : G_{K_n} \longrightarrow \mathbb{F}_{p^n}^\times$ by the property that $g(\pi_n) = [\omega_n(g)]\pi_n$ for all $g \in G_{K_n}$; here $[\cdot] : \mathbb{F}_{p^n}^\times \longrightarrow K_n^\times$ denotes the Teichmuller lift.

Lemma 1.2.3. *The character ω_n defined above is independent of the choice of π_n.*

Proof. Suppose $\pi_n' = \zeta\pi_n$ for some $(p^n-1)^{th}$ root of unity ζ. Since $\zeta \in K_n$, we have $g(\pi_n') = \zeta g(\pi_n) = [\omega_n(g)]\pi_n'$ for $g \in G_{K_n}$; the result follows. □

Example 1.2.4. Suppose that $K = \mathbb{Q}_p$ and $\varpi = -p$. Then ω_1 is just the reduction modulo p of the cyclotomic character $\chi_p : G_{\mathbb{Q}_p} \longrightarrow \mathbb{Z}_p^\times$, defined by the property that $g(\zeta_{p^m}) = \zeta_{p^m}^{\chi_p(g)}$. To see this, observe that $\zeta_p - 1$ has norm p and so

$$\frac{p}{(\zeta_p-1)^{p-1}} = \prod_{i=1}^{p-1} \frac{\zeta_p^i - 1}{\zeta_p - 1} = \prod_{i=1}^{p-1}\sum_{j=0}^{i-1} \zeta_p^j \equiv \prod_{i=1}^{p-1} i = (p-1)! \equiv -1 \pmod{(\zeta_p - 1)}$$

by Wilson's theorem. Thus $(\zeta_p - 1) \equiv \pi_1$, and so

$$\omega_1(g) \equiv \frac{g(\pi_1)}{\pi_1} \equiv \frac{g(\zeta_p-1)}{\zeta_p-1} = \frac{\zeta_p^{\overline{\chi_p}(g)}-1}{\zeta_p-1} = \sum_{j=0}^{\overline{\chi_p}(g)-1} \zeta_p^j = \overline{\chi_p}(g)$$

as required.

Observe that for $h \in G_K$,

$$[\omega_n(hgh^{-1})] = \frac{hgh^{-1}(\pi_n)}{\pi_n} = h\frac{g(h^{-1}\pi_n)}{h^{-1}\pi_n} = h[\omega_n(g)]$$

by the above lemma. Thus $\omega_n(hgh^{-1}) = \omega_n^{p^j}(g)$, where the integer j is in the range $0 \leq j < n$, and in fact corresponds to the image of h in the quotient G_K/G_{K_n} under the isomorphism $G_K/G_{K_n} \cong \mathbb{Z}/n\mathbb{Z}$.

The character ω_n has exact order $(p^n - 1)$; moreover if d divides n then we can take $\pi_d = \pi_n^{\frac{p^n-1}{p^d-1}}$ and so $\omega_d|_{G_{K_n}} = \omega_n^{\frac{p^n-1}{p^d-1}}$. Since $K^t = \bigcup_{n=1}^{\infty} K^{nr}(\pi_n)$, every character of $\mathrm{Gal}(K^t/K^{nr})$ is the restriction to I_K of ω_n^h for appropriate integers n, h. The smallest possible n with this property will be called the level of the character; we then choose h to be primitive for n, meaning that $1 \leq h \leq p^n - 2$ and h is not divisible by $\frac{p^n-1}{p^d-1}$ for any $d < n$ with d dividing n.

We are now in a position to prove the following result on absolutely irreducible k_L-valued representations of G_K.

Proposition 1.2.5. *Let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be an absolutely irreducible representation. Then there is an h, primitive for n, such that $\overline{\rho}$ is isomorphic to $Ind_{G_{K_n}}^{G_K}(\omega_n^h)$ up to unramified twist.*

This is essentially the statement of Theorem 1.2.1 of [2]. We give a proof here for convenience.

Proof. As P_K is pro-p, its image in $GL_n(k_L)$ is contained in a Sylow p-subgroup, which we can take without loss of generality to be the upper triangular unipotent matrices. In particular we see that $\overline{V}^{P_K} \neq 0$, where $\overline{V} \cong (k_L)^n$ denotes the underlying space of the representation $\overline{\rho}$. But since P_K is normal in G_K, $\overline{V}^{P_K}$ is G_K-invariant, and so $\overline{V}^{P_K} = \overline{V}$ as $\overline{\rho}$ is irreducible; that is, P_K acts trivially.

Now since I_K/P_K is abelian of pro-order prime to p, the restriction of $\overline{\rho}$ to I_K splits as a direct sum of characters after extending the field k_L if necessary. Let $\overline{\chi}$

be one such character, and say $\overline{\chi}$ is of level m in the sense of the discussion above. Then $\overline{\chi}$ extends to G_{K_m} and $\overline{\chi}$ occurs in $\overline{\rho}|_{G_{K_m}}$. By Frobenius reciprocity, $Ind_{G_{K_m}}^{G_K}(\overline{\chi})$ occurs in $\overline{\rho}$; since $\overline{\rho}$ is irreducible it then follows that $m = n$ and $\overline{\rho} = Ind_{G_{K_m}}^{G_K}(\overline{\chi})$, as required. □

Remark 1.2.6. Conversely, every representation constructed in this way is indeed irreducible. This follows from Mackey's criterion, together with the observation made above that $\omega_n(hgh^{-1}) = \omega_n^{p^j}(g)$ for an integer j depending only on the image of h in G_K/G_{K_n}.

Corollary 1.2.7. *Let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be an irreducible representation. Then $\overline{\rho}$ has a lift to $\mathcal{O}_L$.*

Proof. This is clear from the above proposition, together with the fact that both unramified characters and characters of the form ω_n^h lift to $\mathcal{O}_L$. □

Remark 1.2.8. It seems to be as yet unknown, though conjectured, whether an arbitrary representation may be lifted in this way. The difficulty lies in being able to lift classes in various Galois cohomology groups and we will not say more about the problem here; however, the reader is advised to consult [27], especially chapter 2, for more details on some results in this direction.

Chapter 2

Crystalline representations and Fontaine-Laffaille modules

In this chapter, we discuss crystalline Galois representations "with coefficients", define the notion of a crystalline representation over a ring $A \in \mathcal{C}_L$, and show that this is a deformation condition. We then review the theory of Fontaine-Laffaille modules, which gives us a way to analyse crystalline representations, at least in the situation where the base field K is unramified over $\mathbb{Q}_p$ and the Hodge-Tate weights lie in the Fontaine-Laffaille range. We finish by giving some examples of these constructions, as well as collecting various results on these modules and their extensions which will be needed to establish the results in the following chapter.

2.1 Crystalline representations with coefficients

Let K be a finite extension of $\mathbb{Q}_p$, and L a subfield of the algebraic closure $\overline{\mathbb{Q}_p}$ containing the images of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$. For simplicity we will focus on the case when K is unramified over $\mathbb{Q}_p$, though much of what we say in this section can be done more generally.

2.1.1 Crystalline in characteristic zero

We first review the notion of L-valued crystalline representations of G_K. For a more thorough summary of much of what we discuss here, see [8].

Admissible representations

Let B be a topological algebra over $\mathbb{Q}_p$ which is an integral domain and furnished with a (continuous) $\mathbb{Q}_p$-linear action of G_K such that B^{G_K} is identified with a finite extension of $\mathbb{Q}_p$ contained within K. Extend the action of G_K to $\mathrm{Frac}(B)$ and suppose additionally that $B^{G_K} = \mathrm{Frac}(B)^{G_K}$. We then make the following definition.

Definition 2.1.1. B is $(\mathbb{Q}_p, G_K)$*-regular* if every G_K-stable $\mathbb{Q}_p$-line $\mathbb{Q}_p \cdot b \subseteq B$ has $b \in B^\times$ (or $b = 0$).

Example 2.1.2. Let $\mathbb{C}_K$ denote the completion of the algebraic closure $\overline{K}$. It is standard that $\mathbb{C}_K$ is itself algebraically closed. We let G_K act on $\mathbb{C}_K$ by continuity. Put

$$B_{HT} = \bigoplus_{r \in \mathbb{Z}} \mathbb{C}_K \cdot x^r = \mathbb{C}_K[x, x^{-1}]$$

and let G_K act on the symbol x via the cyclotomic character. By the Tate-Sen theorem (Theorem 2.2.7 in [8]), $B_{HT}^{G_K} = K$, and the verification that B_{HT} is $(\mathbb{Q}_p, G_K)$-regular is then straightforward.

The most important example for the purposes of this thesis will be the ring B_{cris}, which we will discuss shortly.

Let $\mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ denote the collection of finite dimensional $\mathbb{Q}_p$-vector spaces with a continuous $\mathbb{Q}_p$-linear action of G_K, and let $\mathrm{Mod}_{B^{G_K}}$ denote the set of finite dimensional vector spaces over B^{G_K}. Given any $(\mathbb{Q}_p, G_K)$-regular $\mathbb{Q}_p$-algebra B, we obtain a (covariant) functor

$$D_B : \mathrm{Rep}_{\mathbb{Q}_p}(G_K) \longrightarrow \mathrm{Mod}_{B^{G_K}}$$

sending a module V to $(B \bigotimes_{\mathbb{Q}_p} V)^{G_K}$. By Theorem 5.2.1 of [8], we always have $\dim_{B^{G_K}}(D_B(V)) \leq \dim_{\mathbb{Q}_p}(V)$.

Definition 2.1.3. $V \in \mathrm{Rep}_{\mathbb{Q}_p}(G_K)$ is B-*admissible* if $\dim_{B^{G_K}}(D_B(V)) = \dim_{\mathbb{Q}_p}(V)$. The full subcategory of B-admissible representations of G_K will be denoted as $\mathrm{Rep}^B_{\mathbb{Q}_p}(G_K)$.

The following straightforward properties of B-admissible representations are established in Theorem 5.2.1 of [8].

Proposition 2.1.4. *$Rep^B_{\mathbb{Q}_p}(G_K)$ is a full subcategory of $Rep_{\mathbb{Q}_p}(G_K)$, and the functor $D_B : Rep^B_{\mathbb{Q}_p}(G_K) \longrightarrow Mod_{B^{G_K}}$ is exact and faithful. The category $Rep^B_{\mathbb{Q}_p}(G_K)$ is stable under duals, tensors, direct sums, and subquotients.*

Now let $\mathrm{Rep}_L(G_K)$ denote the collection of finite rank L-modules with a continuous L-linear action of G_K. As above, we get a (covariant) functor, denoted with some abuse of notation as simply

$$D_B : \mathrm{Rep}_L(G_K) \longrightarrow \mathrm{Mod}_{B^{G_K} \otimes_{\mathbb{Q}_p} L}$$

and we let $\mathrm{Rep}^B_L(G_K)$ denote the full subcategory of $V \in \mathrm{Rep}_L(G_K)$ for which $D_B(V)$ is free over $B^{G_K} \bigotimes_{\mathbb{Q}_p} L$ with $\mathrm{rk}_{B^{G_K} \otimes_{\mathbb{Q}_p} L}(D_B(V)) = \dim_L(V)$.

Remark 2.1.5. Since L contains the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, we obtain an isomorphism $K \bigotimes_{\mathbb{Q}_p} L \cong \bigoplus_\sigma L$ by sending $x \otimes y$ to the sequence $(\sigma(x) \cdot y)_\sigma$, analogously to Lemma 2.5.1 below. We will denote by e_σ the element of $K \bigotimes_{\mathbb{Q}_p} L$

which corresponds under this isomorphism to the element with a 1 in the σ component and a 0 in all other components.

Example 2.1.6. Suppose $B = B_{HT}$. A B_{HT}-admissible representation will be called "Hodge-Tate" and the category of such objects will be denoted $\mathrm{Rep}_L^{HT}(G_K)$. For any $V \in \mathrm{Rep}_L^{HT}(G_K)$, we have that

$$(B_{HT} \bigotimes_{\mathbb{Q}_p} V)^{G_K} = \bigoplus_{r \in \mathbb{Z}} (\mathbb{C}_K(\chi_p^r) \bigotimes_{\mathbb{Q}_p} V)^{G_K}$$

is free over $K \bigotimes_{\mathbb{Q}_p} L$, of rank $\dim_L(V)$, and so for each embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, there must exist precisely $\dim_L(V)$ values of r (counted with multiplicity) for which $e_\sigma \cdot ((\mathbb{C}_K(\chi_p^{-r}) \bigotimes_{\mathbb{Q}_p} V)^{G_K}) \neq 0$. Such an r is called a *labelled Hodge-Tate weight* for V (with label σ), and the multiplicity of the labelled Hodge-Tate weight r is $\dim_L(e_\sigma \cdot ((\mathbb{C}_K(\chi_p^{-r}) \bigotimes_{\mathbb{Q}_p} V)^{G_K}))$. The multiset of labelled Hodge-Tate weights for V with label σ will be denoted $HT_\sigma(V)$.

Remark 2.1.7. Note that in the definition of Hodge-Tate weights, we are forced to pick a sign convention - we have the chosen the one for which the cyclotomic character χ_p has all labelled Hodge-Tate weights equal to $+1$.

Construction of B_{dR} and B_{cris}

We briefly sketch a construction of the $(\mathbb{Q}_p, G_K)$-regular $\mathbb{Q}_p$-algebras B_{dR} and B_{cris}. A good review of the properties of these rings can be found in [15], section 2.2, on which this sketch is based. See also [8] for more details, and [18] for a complete description of the constructions.

Let $\mathcal{O}_{\mathbb{C}_K}$ denote the ring of integers of $\mathbb{C}_K$, and consider the inverse limit

$$R = \varprojlim_{n \in \mathbb{N}} \mathcal{O}_{\mathbb{C}_K}/p$$

where the transition maps are (arithmetic) Frobenius ϕ. This is a perfect ring of characteristic p with a natural G_K-action, and so one associates to it the ring of Witt vectors $W(R)$. The actions of G_K and ϕ lift to $W(R)$.

Lemma 2.1.8. *The natural projection*

$$W(R) \longrightarrow R \longrightarrow \mathcal{O}_{\mathbb{C}_K}/p$$

onto the first component lifts to a G_K-equivariant surjection $\theta : W(R) \longrightarrow \mathcal{O}_{\mathbb{C}_K}$. The kernel of θ is principal, and θ extends to a map $\theta_{\mathbb{Q}} : W(R)[\frac{1}{p}] \longrightarrow \mathbb{C}_K$.

Proof. See [8], Lemma 4.4.1 and Proposition 4.4.3. □

We define B_{dR}^+ as the completion of $W(R)[\frac{1}{p}]$ with respect to the ideal $\mathrm{Ker}(\theta_{\mathbb{Q}})$. This is a complete discrete valuation ring, with residue field $\mathbb{C}_K$. We define $B_{dR} = \mathrm{Frac}(B_{dR}^+)$. Since θ is G_K-equivariant, we get a (continuous with respect to the $\mathrm{Ker}(\theta_{\mathbb{Q}})$-adic topology) action of G_K on B_{dR}^+ and B_{dR}.

Letting $(\epsilon_j)_{j \in \mathbb{N}} \in \mathcal{O}_{\mathbb{C}_K}$ denote a p-power compatible sequence of primitive $(p^j)^{th}$ roots of unity gives rise to elements $\epsilon \in R$ and $[\epsilon] \in W(R)$ (where $[\cdot]$ denotes the Teichmuller lift). Since $\theta([\epsilon]) = 1$, the sum

$$t = log([\epsilon]) = \sum_{r=1}^{\infty} (-1)^{r+1} \frac{([\epsilon] - 1)^r}{r}$$

converges in B_{dR}^+.

Proposition 2.1.9. *1. The element t defined above is a uniformiser of B_{dR}^{+}, and G_K acts on t as $g(t) = \chi_p(g)t$ (where χ_p is the cyclotomic character). The filtration defined on B_{dR} via $Fil^i(B_{dR}) = t^i B_{dR}^{+}$ for $i \in \mathbb{Z}$ leads to a natural identification of $gr^{\bullet}(B_{dR})$ and B_{HT}.*

2. B_{dR}^{+} naturally has the structure of a K-algebra.

3. B_{dR} is $(\mathbb{Q}_p, G_K)$-regular in the sense of Definition 2.1.1, and $B_{dR}^{G_K} = K$.

Proof. The proof of all of these statements can be found in [8]. In particular, the first statement follows from Proposition 4.4.8, the second from Lemma 4.4.10, and the third from Theorem 4.4.13 together with the fact that B_{dR} is a field. □

We thus obtain categories $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{dR}}(G_K)$ and $\mathrm{Rep}_{L}^{B_{dR}}(G_K)$ (denoted more simply as $\mathrm{Rep}_{\mathbb{Q}_p}^{dR}(G_K)$ and $\mathrm{Rep}_{L}^{dR}(G_K)$, respectively) in the sense of Definition 2.1.3. Such objects are called "deRham" representations. It follows from the proposition above that objects in the image of the functor $D_{B_{dR}}$ have a natural filtration (so we interpret $D_{B_{dR}}$ as landing inside the category $\mathrm{Fil}(K) \bigotimes_{\mathbb{Q}_p} L$ of filtered modules over K whose endormorphism rings have the structure of an L-algebra), and that all deRham representations are Hodge-Tate (in fact, $\mathrm{gr}^{\bullet}(D_{B_{dR}}(V)) = D_{B_{HT}}(V)$ for any G_K-representation V).

We now turn to B_{cris}. Letting $(p_j)_{j \in \mathbb{N}} \in \mathcal{O}_{\mathbb{C}_K}$ denote a p-power compatible sequence of $(p^j)^{th}$ roots of p gives rise to elements $\underline{p} \in R$ and $[\underline{p}] \in W(R)$.

Lemma 2.1.10. *The element $\zeta = [\underline{p}] - p$ generates the ideal $Ker(\theta)$ of $W(R)$.*

Proof. See [8], Proposition 4.4.3 (part 1). □

We let A_{cris} denote the p-adic completion of the divided power envelope of $W(R)$ with respect to the ideal generated by ζ; the Frobenius ϕ then extends continuously to A_{cris} by [8], Lemma 9.1.7. Observe also that since $[\epsilon] - 1 \in \zeta \cdot W(R)$, we can interpret t as an element of A_{cris}. Define $B_{cris} = A_{cris}[\frac{1}{t}]$. As for B_{dR}, the ring B_{cris} is equipped with a (continuous) action of G_K.

Proposition 2.1.11. *1. The Frobenius ϕ acts on t via $\phi(t) = pt$. The action of Frobenius can be extended to B_{cris} by setting $\phi(t^{-1}) = p^{-1}t^{-1}$, and is injective and commutes with the action of G_K.*

2. Assuming K is unramified over $\mathbb{Q}_p$, B_{cris} naturally has the structure of a K-algebra, and there is an injection $A_{cris} \hookrightarrow B_{dR}^{+}$ which extends to $B_{cris} \hookrightarrow B_{dR}$. We can therefore give B_{cris} the filtration induced from that on B_{dR}.

3. B_{cris} is $(\mathbb{Q}_p, G_K)$-regular in the sense of Definition 2.1.1, and assuming K is unramified over $\mathbb{Q}_p$, we have that $B_{cris}^{G_K} = K$.

Proof. The proof of all of these statements can be found in [8]. In particular, the first statement follows from Theorem 9.1.8, the second from Theorem 9.1.5, and the third from Proposition 9.1.6. □

We thus obtain categories $\mathrm{Rep}_{\mathbb{Q}_p}^{B_{cris}}(G_K)$ and $\mathrm{Rep}_{L}^{B_{cris}}(G_K)$ (denoted more simply as $\mathrm{Rep}_{\mathbb{Q}_p}^{cris}(G_K)$ and $\mathrm{Rep}_{L}^{cris}(G_K)$, respectively) in the sense of Definition 2.1.3. Such objects are called "crystalline" representations. It follows from the proposition above that when K is unramified over $\mathbb{Q}_p$, the image of the functor $D_{B_{cris}}$ lands inside the category $MF_K^{\phi} \bigotimes_{\mathbb{Q}_p} L$ as defined below. Moreover, all crystalline representations are deRham (in fact, $D_{B_{cris}}(V) = D_{B_{dR}}(V)$ as filtered modules if V is crystalline - see [8], Proposition 9.1.9), and hence Hodge-Tate.

Filtered ϕ-modules

We now present a category of semilinear algebra data which corresponds to characteristic zero crystalline representations. We let τ denote the endormophism $\phi \otimes 1$ of $K \bigotimes_{\mathbb{Q}_p} L$, where ϕ denotes (arithmetic) Frobenius on K (recall K is unramified over $\mathbb{Q}_p$). For more details, see [7] or [21].

Definition 2.1.12. 1. A *filtered ϕ-module over K* is a finite dimensional K-vector space D, together with a Frobenius-semilinear automorphism ϕ of D and a decreasing, separated, exhaustive filtration of D by K-subspaces D^i ($i \in \mathbb{Z}$). We denote the category of such objects as MF_K^ϕ; morphisms in this category are K-linear ϕ-equivariant homomorphisms which respect the filtrations.

2. A *filtered ϕ-module over K with coefficients in L* is a free finite rank $K \bigotimes_{\mathbb{Q}_p} L$-module D, together with a τ-semilinear automorphism ϕ of D and a decreasing, separated, exhaustive filtration of D by $K \bigotimes_{\mathbb{Q}_p} L$-submodules D^i ($i \in \mathbb{Z}$). We denote the category of such objects as $MF_K^\phi \bigotimes_{\mathbb{Q}_p} L$; morphisms in this category are $K \bigotimes_{\mathbb{Q}_p} L$-linear ϕ-equivariant homomorphisms which respect the filtrations.

Note that giving $D \in MF_K^\phi \bigotimes_{\mathbb{Q}_p} L$ is equivalent to giving $D \in MF_K^\phi$ together with a morphism $L \hookrightarrow \mathrm{End}_{MF_K^\phi}(D)$ making D free over $K \bigotimes_{\mathbb{Q}_p} L$. Conversely, we get a functor $MF_K^\phi \bigotimes_{\mathbb{Q}_p} L \longrightarrow MF_K^\phi$ by forgetting the L-structure.

We are primarily interested in certain full subcategories of these (the so called weakly admissible modules), since these are the ones which correspond to L-valued crystalline representations of G_K. To make this definition we must define the Hodge and Newton numbers associated to a filtered ϕ-module D over K with coefficients in L. The definitions are exactly the same as in [8], considering such a module as having coefficients in K by ignoring the L-structure. We briefly recall these definitions here.

Definition 2.1.13. Let $D \in MF_K^\phi$. The *Hodge number* of D is

$$t_H(D) = \sum_{i \in \mathbb{Z}} i \cdot \dim_K(\mathrm{gr}^i(D)).$$

Definition 2.1.14.

1. For $D \in MF_K^\phi$, the *Hodge-Tate weights* of D are those integers i for which $\mathrm{gr}^i(D) \neq 0$. The *multiplicity* of a Hodge-Tate weight i is the number $\dim_K(\mathrm{gr}^i(D))$. We write $HT(D)$ for the multiset of Hodge-Tate weights of D, counted with multiplicity.

2. For $D \in MF_K^\phi \bigotimes_{\mathbb{Q}_p} L$ and an embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, the *labelled Hodge-Tate weights* of D with label σ are those integers i for which $e_\sigma \cdot \mathrm{gr}^i(D) \neq 0$. Here $e_\sigma \in K \bigotimes_{\mathbb{Q}_p} L$ is as in Remark 2.1.5. The *multiplicity* of a labelled Hodge-Tate weight i with label σ is the number $\dim_K(e_\sigma \cdot \mathrm{gr}^i(D))$. We write $HT_\sigma(D)$ for the multiset of labelled Hodge-Tate weights of D with label σ, counted with multiplicity.

To define the Newton number, we need the following result. Recall that we continue to assume that K is unramified over $\mathbb{Q}_p$.

Theorem 2.1.15. *Let $D \in MF_K^\phi$. After extending scalars to the completion $\hat{K^{nr}}$ of K^{nr}, the object $D \bigotimes_K \hat{K^{nr}}$ splits as a (finite) direct sum of simple objects of the category $Mod^\phi_{\hat{K^{nr}}}$ of finite dimensional $\hat{K^{nr}}$-vector spaces together with a Frobenius-semilinear automorphism ϕ. The simple objects Δ_α of $Mod^\phi_{\hat{K^{nr}}}$ are indexed by $\alpha \in \mathbb{Q}$ and are given in the following definition.*

Proof. See the main result of [24], chapter II, section 4.1. Some more discussion of this result may be found in [8], Theorem 8.1.4. □

Definition 2.1.16. Let $\alpha = \frac{s}{r} \in \mathbb{Q}$ with s, r coprime and $r > 0$. We define Δ_α as the quotient of the twisted polynomial ring $\hat{K^{nr}}\{x\}$ by the left ideal generated by $x^r - p^s$; here, the twisted polynomial ring $\hat{K^{nr}}\{x\}$ is defined by the property that $x \cdot c = \phi(c) \cdot x$ for $x \in \hat{K^{nr}}$. The action of ϕ on Δ_α is as left multiplication by x.

For $D \in MF_K^\phi$ and $\alpha \in \mathbb{Q}$, we will write $\hat{D}(\alpha)$ for the component of $D \bigotimes_K \hat{K^{nr}}$ corresponding to α; that is, if $D \bigotimes_K \hat{K^{nr}} \cong \bigoplus_{\beta \in \mathbb{Q}} \Delta_\beta^{e_\beta}$ then $\hat{D}(\alpha) = \Delta_\alpha^{e_\alpha}$. Of course for a given D, $\hat{D}(\alpha) = 0$ for all but finitely many α.

We are now in a position to define the Newton number.

Definition 2.1.17. Let $D \in MF_K^\phi$. The *Newton number* of D is

$$t_N(D) = \sum_{\alpha \in \mathbb{Q}} \alpha \cdot \dim_{\hat{K^{nr}}}(\hat{D}(\alpha)).$$

Remark 2.1.18. If $\dim_K(D) = 1$ and ϕ acts on a basis element as multiplication by $\lambda \in K$ then $t_N(D) = v_p(\lambda)$, where v_p is the p-adic valuation, normalised so that $v_p(p) = 1$. In particular, $v_p(\lambda)$ is independent of the choice of basis. Given any $D \in MF_K^\phi$ of dimension n, it follows by Proposition 8.1.9 of [8] that $t_N(D) = t_N(\Lambda^n(D))$, which gives an easier way to calculate Newton numbers.

We have thus associated to $D \in MF_K^\phi \bigotimes_{\mathbb{Q}_p} L$ the quantities $t_H(D)$ and $t_N(D)$, and proceed to make the following definition.

Definition 2.1.19. A filtered ϕ-module $D \in MF_K^\phi \bigotimes_{\mathbb{Q}_p} L$ is *weakly admissible* if $t_H(D) = t_N(D)$ and furthermore if for all subobjects $D' \subseteq D$ in $MF_K^\phi \bigotimes_{\mathbb{Q}_p} L$ we have that $t_H(D') \leq t_N(D')$. The full subcategory of all such objects is denoted $MF_K^{\phi, w.a.} \bigotimes_{\mathbb{Q}_p} L$.

Remark 2.1.20. By Proposition 3.1.1.5 of [7], this is equivalent to checking that $t_H(D') \leq t_N(D')$ for all subobjects $D' \subseteq D$ in MF_K^{ϕ}.

Example 2.1.21. We classify objects $D \in MF_K^{\phi,w.a.} \bigotimes_{\mathbb{Q}_p} L$ of rank 1 up to isomorphism. D is specified by its collection $(n_\sigma)_\sigma$ of labelled Hodge-Tate weights, together with an element of $(K \bigotimes_{\mathbb{Q}_p} L)^\times$ determined up to semilinear conjugation by τ specifying the action of ϕ; this set is in canonical bijection with $L^\times$ by sending $(\phi_\sigma)_\sigma$ (with $\phi_\sigma \in L$) to $\prod_\sigma \phi_\sigma$. With some abuse of notation, we will denote the element of $L^\times$ thus constructed as simply ϕ (or ϕ_D if we want to emphasise the dependence on D).

From the definitions we see that $t_H(D) = [L : K] \sum_\sigma n_\sigma$ and $t_N(D) = v_K(N_{L/K}(\phi))$ where v_K is the p-adic valuation on K normalised so that $v_K(p) = 1$, $N_{L/K}$ denotes the relative norm, and we regard K as a subfield of L using any of the embeddings σ. Since D has rank 1, there are no proper subobjects to consider, so we only need to ensure that $t_H(D) = t_N(D)$. Putting $\lambda = p^{-\sum_\sigma n_\sigma}\phi$, we see that D is specified up to isomorphism by the collection $(n_\sigma)_\sigma$ together with a unit $\lambda \in \mathcal{O}_L^\times$.

For a similar classification of rank 2 objects, see [21].

Proposition 2.1.22. *The functor $D_{B_{cris}} = D_{cris}$ is an equivalence of abelian categories*

$$D_{cris} : Rep_L^{cris}(G_K) \longrightarrow MF_K^{\phi,w.a.} \bigotimes_{\mathbb{Q}_p} L.$$

If $V \in Rep_L^{cris}(G_K)$ then $V \bigotimes_{\mathbb{Q}_p} K$ is isomorphic to $D_{cris}(V)$ (non-canonically) as a $K \bigotimes_{\mathbb{Q}_p} L$-module. Additionally, for all $V \in Rep_L^{cris}(G_K)$ we have that as multisets counted with multiplicity, $HT_\sigma(V) = HT_\sigma(D_{cris}(V))$ for each embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$.

Proof. This follows from the standard statement for crystalline representations over $\mathbb{Q}_p$ that "weakly admissible implies admissible" (Theorem A of [11], recalling that

K is unramified over $\mathbb{Q}_p$) together with the observation that giving an object of either of the above two categories is equivalent to giving a corresponding $\mathbb{Q}_p$-valued object D as well as an algebra homomorphism $L \longrightarrow \mathrm{End}(D)$. For more details, consult [23], especially Proposition 1.3.4. Note that for most of this thesis we will in fact only be using this result when the Hodge-Tate weights (defined below) lie in a certain restricted range (and K is unramified over $\mathbb{Q}_p$), in which case the result in fact follows from the weaker statement of [19] (main Theorem). $\square$

Given two crystalline (or even Hodge-Tate) representations ρ_1, ρ_2 and an embedding σ, we write $HT_\sigma(\rho_1) > HT_\sigma(\rho_2)$ if there is an integer i such that all elements of $HT_\sigma(\rho_1)$ are greater than or equal to i, and all elements of $HT_\sigma(\rho_2)$ are strictly less than i. We also let $\mathrm{Ext}^1_{cris,L}(\rho_2, \rho_1)$ denote the set of extensions of ρ_2 by ρ_1 which are crystalline.

Proposition 2.1.23. *Let V_i ($i = 1,2$) be crystalline L-representations of G_K of dimension n_i such that for each σ, $HT_\sigma(V_1) > HT_\sigma(V_2)$. Then $Ext^1_{cris,L}(V_2, V_1)$ is of dimension $n_1 n_2 [K : \mathbb{Q}_p]$ over L.*

Remark 2.1.24. This is a standard result for $\mathbb{Q}_p$-representations of G_K; we give the analogous proof for L-representations for convenience.

Proof. For $i = 1, 2$, let V_i correspond to the weakly admissible filtered ϕ-module D_i with τ-semilinear action ϕ_i. Any $D \in \mathrm{Ext}^1_{MF}(D_1, D_2)$ is isomorphic to $D_1 \bigoplus D_2$ as a filtered $K \bigotimes_{\mathbb{Q}_p} L$-module. Moreover, after extending a $K \bigotimes_{\mathbb{Q}_p} L$-basis of D_2 to D, we see that a τ-semilinear action ϕ on D gives rise to an extension $D \in \mathrm{Ext}^1_{MF}(D_1, D_2)$ precisely when it takes the form

$$\phi = \begin{pmatrix} \phi_2 & * \\ 0 & \phi_1 \end{pmatrix}$$

with respect to this basis, and so the set of allowable ϕ corresponds to $(K \bigotimes_{\mathbb{Q}_p} L)^{n_1 n_2} = L^{n_1 n_2 [K:\mathbb{Q}_p]}$. So it suffices to show that any such D is weakly admissible.

By Propositions 8.1.2 and 8.1.9 of [8], t_N and t_H are additive in short exact sequences, so $t_N(D) = t_H(D)$. Suppose D' is a subobject of D, and put $D'_1 = D_1 \cap D'$, $D'_2 = (D' + D_1)/D_1$. It is straightforward that D'_i is a subobject of D_i; we show that the sequence

$$0 \longrightarrow D'_1 \longrightarrow D' \longrightarrow D'_2 \longrightarrow 0$$

is exact in the category $MF_K^{\phi, w.a.} \bigotimes_{\mathbb{Q}_p} L$. This will suffice since then

$$t_N(D') = t_N(D'_1) + t_N(D'_2) \geq t_H(D'_1) + t_H(D'_2) = t_H(D')$$

and so D is weakly admissible.

The given sequence is clearly exact in the category of $K \bigotimes_{\mathbb{Q}_p} L$-modules, so it remains to prove it is exact on filtrations; or in other words that for every k, σ, the sequence

$$0 \longrightarrow (D_1)^k_\sigma \cap D'_\sigma \longrightarrow D^k_\sigma \cap D'_\sigma \longrightarrow (D_2)^k_\sigma \cap (D'_2)_\sigma \longrightarrow 0$$

is exact. By the hypothesis, there is some integer j (depending on σ) such that $(D_1)^j_\sigma = 0$ and $(D_2)^j_\sigma = (D_2)_\sigma$. Thus if $k \leq j$ then the above sequence reads

$$0 \longrightarrow (D_1)^k_\sigma \cap D'_\sigma \longrightarrow D^k_\sigma \cap D'_\sigma \longrightarrow \frac{D'_\sigma + (D_1)_\sigma}{(D_1)_\sigma} \longrightarrow 0$$

while if $k \geq j$ it reads

$$0 \longrightarrow D^k_\sigma \cap D'_\sigma \longrightarrow \frac{(D'_\sigma + (D_1)_\sigma) \cap (D^k_\sigma + (D_1)_\sigma)}{(D_1)_\sigma} \longrightarrow 0.$$

Both of these follow from rank calculations:

1. In the case when $k \leq j$, the above sequence is exact whenever $\dim_L(D^k_\sigma \cap D'_\sigma) = \dim_L((D_1)^k_\sigma \cap D'_\sigma) + \dim_L(\frac{D'_\sigma+(D_1)_\sigma}{(D_1)_\sigma})$; or in other words, if

   $$\dim_L(D'_\sigma + (D_1)_\sigma) + \dim_L((D_1)_\sigma \cap D'_\sigma \cap D^k_\sigma) = \dim_L((D_1)_\sigma) + \dim_L(D^k_\sigma \cap D'_\sigma).$$

   This is equivalent to asking that $(D_1)_\sigma + D'_\sigma = (D_1)_\sigma + (D^k_\sigma \cap D'_\sigma)$. Since $(D_2)^k_\sigma = (D_2)_\sigma$ and $(D'_2)_\sigma \subseteq (D_2)_\sigma$, we conclude that

   $$\frac{(D_1)_\sigma + (D^k_\sigma \cap D'_\sigma)}{(D_1)_\sigma} = \frac{(D_1)_\sigma + D'_\sigma}{(D_1)_\sigma}$$

   which gives the result.

2. In the case when $k \geq j$, the sequence is exact whenever $\dim_L(D^k_\sigma \cap D'_\sigma) = \dim_L(\frac{(D'_\sigma+(D_1)_\sigma)\cap(D^k_\sigma+(D_1)_\sigma)}{(D_1)_\sigma})$; or in other words, if

   $$\dim_L(D^k_\sigma \cap D'_\sigma) + \dim_L((D_1)_\sigma)$$
   $$= \dim_L((D'_\sigma + (D_1)_\sigma)) + \dim_L(D^k_\sigma + (D_1)_\sigma) - \dim_L(D'_\sigma + (D_1)_\sigma + D^k_\sigma).$$

   But since $(D_1)_\sigma \cap D^k_\sigma = 0$, we have

$$\dim_L(D'_\sigma + (D_1)_\sigma + D^k_\sigma)$$
$$= \dim_L(D'_\sigma) + \dim_L((D_1)_\sigma) + \dim_L(D^k_\sigma) - \dim_L(D^k_\sigma \cap D'_\sigma) - \dim_L((D_1)_\sigma \cap D'_\sigma)$$

and so

$$\dim_L((D'_\sigma + (D_1)_\sigma)) + \dim_L(D^k_\sigma + (D_1)_\sigma) - \dim_L(D'_\sigma + (D_1)_\sigma + D^k_\sigma)$$
$$= \dim_L(D^k_\sigma \cap D'_\sigma) + \dim_L((D_1)_\sigma)$$

which again gives the result.

Thus D is weakly admissible as required. □

Crystalline characters and local algebraicity

As an exercise in the above definitions, we provide a brief characterisation of crystalline characters in characteristic zero, following appendix B of [12]. Recall we have normalised the Artin map $r_K : K^\times \hookrightarrow G_K^{ab}$ so that a uniformiser of K maps to a lift of arithmetic Frobenius. We let $\underline{K^\times}$ denote the multiplicative group $\mathbb{G}_m$ over K, thought of as an algebraic group over $\mathbb{Q}_p$ (and likewise for $\underline{L^\times}$).

Definition 2.1.25. A character $\chi : G_K \longrightarrow L^\times$ is *locally algebraic* if there exists a $\mathbb{Q}_p$-homomorphism $\underline{K^\times} \longrightarrow \underline{L^\times}$ of algebraic groups over $\mathbb{Q}_p$ which, on $\mathbb{Q}_p$-points, agrees with the composition $\chi \circ r_K$ in a neighbourhood of 1.

Recall that L contains the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$. Thus for every $\mathbb{Q}_p$-homomorphism $\theta : \underline{K^\times} \longrightarrow \underline{L^\times}$ of algebraic groups over $\mathbb{Q}_p$, there are integers $(n_\sigma)_\sigma$ such that $\theta(x) = \prod_\sigma \sigma(x)^{n_\sigma}$ for $x \in \underline{K^\times}$.

We can now state the following result characterising when the character χ is Hodge-Tate or crystalline.

Proposition 2.1.26. *Let $\chi : G_K \longrightarrow L^\times$ be any character.*

1. *χ is Hodge-Tate if and only if it is locally algebraic.*

2. *χ is crystalline if and only if it is locally algebraic, and additionally the neighbourhood of 1 in Definition 2.1.25 can be taken to be $\mathcal{O}_K^\times$.*

Proof. See appendix B of [12], in particular Proposition B.4. □

Remark 2.1.27. By local class field theory (Proposition 1.2.1), r_K identifies $\mathcal{O}_K^\times$ with I_K^{ab}, and so a crystalline character χ as above is specified by the collection $(n_\sigma)_\sigma$ (which determines $\chi|_{I_K}$) together with a unit $\lambda \in \mathcal{O}_L^\times$ giving the value of χ on any lift of Frobenius; as expected by the classification of rank 1 filtered ϕ-modules in example 2.1.21.

2.1.2 Crystalline in characteristic p

We need a way to discuss whether A-valued representations (for $A \in \mathcal{C}_L$) are crystalline. Taking inspiration from Definition 2.12 of [22], we can make the following definition.

Definition 2.1.28. Let $A \in \mathcal{C}_L$, and suppose $\rho : G_K \longrightarrow GL_n(A)$ is a representation. We say that ρ is *crystalline* with Hodge-Tate weights in some interval $[a, b]$ if there is a crystalline L-representation V of G_K (in the sense of the previous section) with labelled Hodge-Tate weights all contained in the interval $[a, b]$, containing G_K-stable $\mathcal{O}_L$-lattices $T' \subseteq T$, and an $\mathcal{O}_L$-algebra map $A \longrightarrow \mathrm{End}_{\mathcal{O}_L}(\frac{T}{T'})$ such that A^n (with G_K-action given by ρ) is isomorphic as an $A[G_K]$-module to $\frac{T}{T'}$.

Remark 2.1.29. Given an integer $r \geq 0$, consider those $\rho : G_K \longrightarrow GL_n(A)$ such that the crystalline L-representation V as above can additionally be chosen with all labelled Hodge-Tate weights in the range $[0, r]$. Then $\rho \in \mathrm{Rep}_A^{cris, \leq r}(G_K)$ in the notation of the next section.

Definition 2.1.30. Let $\rho : G_K \longrightarrow GL_n(A)$ be a representation, with underlying A-module $V \cong A^n$. We say ρ is *reducible* if there is a non-trivial A-submodule $W \subseteq V$, which is a free direct summand of V and stable under the action of G_K. Otherwise, we say that ρ is *irreducible*.

As in the case of characteristic zero representations, we seek some semilinear algebra data which classifies such representations in much the same way as Proposition 2.1.22. We carry this out in the following section.

2.2 Fontaine-Laffaille modules and crystalline representations in characteristic p

We now introduce various categories of algebraic objects that, in certain cases, allow us to classify crystalline representations in characteristic p in much the same way that one classifies crystalline representations in characteristic zero using $MF_K^{\phi, w.a.} \bigotimes_{\mathbb{Q}_p} L$. The main references for the results in this section are [19] and [22]. We continue to assume that K is a finite unramified extension of $\mathbb{Q}_p$, and that L is a finite extension of $\mathbb{Q}_p$ inside $\overline{\mathbb{Q}_p}$ containing the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$.

Definition 2.2.1. 1. A Fontaine-Laffaille module is a finite length module over $\mathcal{O}_K$ together with a decreasing filtration by $\mathcal{O}_K$-module direct summands M^i

such that $M^0 = M$, $M^p = 0$, and a collection of Frobenius-semilinear maps $\phi_M^i : M^i \longrightarrow M$ such that $\phi_M^i|_{M^{i+1}} = p\phi_M^{i+1}$ for all i, and $M = \sum_i \phi_M^i(M^i)$. The corresponding category is denoted $MF_{tor,\mathcal{O}_K}^{f,p-1}$; morphisms in this category are filtration-preserving $\mathcal{O}_K$-linear maps which are equivariant with respect to the corresponding ϕ_i for all i. When there is no risk of confusion we will write simply ϕ^i in place of ϕ_M^i.

2. A *Fontaine-Laffaille module over A* for $A \in \mathcal{C}_L$ consists of giving an object $M \in MF_{tor,\mathcal{O}_K}^{f,p-1}$ together with a map $\theta : A \longrightarrow \mathrm{End}_{MF_{tor,\mathcal{O}_K}^{f,p-1}}(M)$ that makes M into a free finitely generated module over $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$. A morphism between two such objects is required to additionally preserve the A-structure. We will denote this category as $MF_{tor,\mathcal{O}_K}^{f,p-1} \bigotimes_{\mathbb{Z}_p} A$.

We note also that $MF_{tor,\mathcal{O}_K}^{f,p-1}$ is an abelian category, as follows from [19], Proposition 1.8.

Definition 2.2.2. Let $M \in MF_{tor,\mathcal{O}_K}^{f,p-1}$. A *submodule* of M is an $\mathcal{O}_K$ submodule $N \subseteq M$ given the subspace filtration $N^i = N \cap M^i$ such that $\phi^i(N^i) \subseteq N$ for all i. When M has A-structure for some $A \in \mathcal{C}_L$ we additionally demand that N be a free $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$-module direct summand of M. If M has no submodules then we say M is *irreducible.*

We will have occasion to use various full subcategories of $MF_{tor,\mathcal{O}_K}^{f,p-1}$; in particular the following.

Definition 2.2.3. 1. $MF_{tor,\mathcal{O}_K}^{f,p-1}{}'$ consists of those objects $M \in MF_{tor,\mathcal{O}_K}^{f,p-1}$ which have no non-trivial quotient object N with $N^{p-1} = N$.

2. $MF^{f,p-1}_{tor,\mathcal{O}_K}{}''$ consists of those objects $M \in MF^{f,p-1}_{tor,\mathcal{O}_K}$ which have no non-trivial subobject N with $N^1 = 0$.

3. Let $0 \leq r \leq p-1$. Then $MF^{f,r}_{tor,\mathcal{O}_K}$ consists of those objects $M \in MF^{f,p-1}_{tor,\mathcal{O}_K}$ such that $M^{r+1} = 0$.

We also have the analog of these subcategories for $MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$, where the objects are required additionally to have A-structure in the sense of part 2 of Definition 2.2.1. These analogous full subcategories will be denoted (respectively) as $MF^{f,p-1}_{tor,\mathcal{O}_K}{}' \bigotimes_{\mathbb{Z}_p} A$, $MF^{f,p-1}_{tor,\mathcal{O}_K}{}'' \bigotimes_{\mathbb{Z}_p} A$, and $MF^{f,r}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$.

Remark 2.2.4. $MF^{f,r}_{tor,\mathcal{O}_K} \subseteq MF^{f,p-1}_{tor,\mathcal{O}_K}{}'$ for $0 \leq r \leq p-2$.

We now give some basic facts about Fontaine-Laffaille modules.

Proposition 2.2.5. *Let $M, N \in MF^{f,p-1}_{tor,\mathcal{O}_K}$ and $f \in Hom_{MF^{f,p-1}_{tor,\mathcal{O}_K}}(M, N)$.*

1. *For all i, $M^i \subseteq M$ is a direct summand.*

2. *f is strict with filtrations in the sense that $f(M^i) = f(M) \cap N^i$ for all i.*

Proof. 1. See [22], Corollary 2.6 (ii).

2. See [19], 1.10 (b). □

Definition 2.2.6. Let $A \in \mathcal{C}_L$. For every embedding $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, we denote by e_σ the element of $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$ whose component at σ is 1 and all other components are 0, in the sense of Lemma 2.5.1. If M is a Fontaine-Laffaille module over A, put $M_\sigma = e_\sigma \cdot M$ and $M^i_\sigma = e_\sigma \cdot M^i$ for all i.

Note that for all σ, $\tau(e_\sigma) = e_{\sigma \cdot Frob^{-1}}$; thus if M is a Fontaine-Laffaille module over A of rank n, then $\phi^i(M^i_\sigma) \subseteq M_{\sigma \cdot Frob^{-1}}$ for all i, and $M_{\sigma \cdot Frob^{-1}} = \sum_i \phi^i(M^i_\sigma)$.

Considering M_σ as an A-module in the natural way, M_σ is free over A of rank n, and the M_σ^i are free direct summands. Any i for which $\frac{M_\sigma^i}{M_\sigma^{i+1}} \neq 0$ is called a labelled Hodge-Tate weight for M (with label σ); the multiplicity of the label is $rk_A(\frac{M_\sigma^i}{M_\sigma^{i+1}})$. The multiset of labelled Hodge-Tate weights (counted with multiplicity) of a Fontaine-Laffaille module M over A for any embedding σ will be denoted as $HT_\sigma(M)$.

Remark 2.2.7. The definition of Hodge-Tate weights for Fontaine-Laffaille modules given above is consistent with the contravariant Fontaine-Laffaille functor U_S introduced in Theorem 2.2.8 and the convention adopted that the cyclotomic character has weight $+1$. That is, under this convention, we have that $HT_\sigma(M) = HT_\sigma(U_S(M))$ for any $M \in MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$. For a justification of why this convention is consistent, see Remark 2.4.4.

The main reason for our interest in Fontaine-Laffaille modules is the following fundamental Theorem.

Theorem 2.2.8. *1. There is a contravariant functor*

$$U_S : MF^{f,p-1}_{tor,\mathcal{O}_K} \longrightarrow Rep^f_{\mathbb{Z}_p}(G_K)$$

which is exact, additive, faithful, and length preserving in the sense that $lg_{\mathbb{Z}_p}(U_S(M)) = lg_{\mathcal{O}_K}(M)$ *for all* $M \in MF^{f,p-1}_{tor,\mathcal{O}_K}$ *(here, lg denotes the length of a module). Moreover,* M *has the same invariant factors over* $\mathcal{O}_K$ *as* $U_S(M)$ *does over* $\mathbb{Z}_p$.

2. U_S *is full when restricted to either of the full subcategories* $MF^{f,p-1}_{tor,\mathcal{O}_K}{}'$ *or* $MF^{f,p-1}_{tor,\mathcal{O}_K}{}''$ *of* $MF^{f,p-1}_{tor,\mathcal{O}_K}$.

3. *For any $A \in \mathcal{C}_L$, every object in the essential image of U_S on $MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ is crystalline in the sense of Definition 2.1.28. For any $0 \le r \le p-2$, U_S gives an anti-equivalence of categories between $MF^{f,r}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ and its essential image, denoted as $Rep^{cris,\le r}_A(G_K)$.*

Proof. The first two statements are proved in [19], in particular Theorem 3.3 and Theorem 6.1. An explicit construction of the functor U_S is given in section 2 of [19]. For a proof of the third statement, see [6], Theorem 3.1.3.3. □

Definition 2.2.9. Representations in $\mathrm{Rep}^{cris,\le p-2}_A(G_K)$ (and their associated Fontaine-Laffaille modules) will be said to have Hodge-Tate weights in the *Fontaine-Laffaille range*. We will also refer to such representations as *Fontaine-Laffaille representations.*

Lemma 2.2.10. *For any $A \in \mathcal{C}_L$, a Fontaine-Laffaille module $M \in MF^{f,p-2}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ is irreducible if and only if the associated representation $U_S(M)$ of G_K valued in A is irreducible.*

Proof. This is immediate from the fact that U_S is an (anti-)equivalence. □

For convenience we will sometimes replace the functor U_S above with the covariant version $U(M) = U_S(\mathrm{Hom}(M, \frac{K}{\mathcal{O}_K}\{p-2\}))(2-p)$, defined as follows (see [10]):

1. $\mathrm{Hom}(M, \frac{K}{\mathcal{O}_K}\{p-2\})(2-p)$ has underlying module $\mathrm{Hom}_{\mathcal{O}_K}(M, \frac{K}{\mathcal{O}_K})$.

2. The i^{th} filtration piece of $\mathrm{Hom}(M, \frac{K}{\mathcal{O}_K}\{p-2\})(2-p)$ is $\mathrm{Hom}_{\mathcal{O}_K}(\frac{M}{M^{p-1-i}}, \frac{K}{\mathcal{O}_K})$.

3. For $f \in \mathrm{Hom}_{\mathcal{O}_K}(\frac{M}{M^{p-1-i}}, \frac{K}{\mathcal{O}_K})$, define $\phi^i(f)$ on M by setting $\phi^i(f)(\phi^j(m^j)) = p^{p-2-i-j}\mathrm{Frob}(f(m^j))$ (for $m^j \in M^j$), and extending τ-semilinearly.

Remark 2.2.11. It is straightforward to check that this is well defined, and makes $\mathrm{Hom}(M, \frac{K}{\mathcal{O}_K}\{p-2\})(2-p)$ into a Fontaine-Laffaille module.

We now give some simple results on the Fontaine-Laffaille functor that will be needed later.

Lemma 2.2.12. *Let M be a rank n Fontaine-Laffaille module over A. Then $U_S(M)$ is free over A of rank n.*

Proof. $\mathcal{O}_K$ is free and hence faithfully flat over $\mathbb{Z}_p$. The result thus follows from the fact that $U_S(M) \bigotimes_{\mathbb{Z}_p} \mathcal{O}_K \cong M$ as $\mathcal{O}_K$-modules. □

Lemma 2.2.13. *Let $A \longrightarrow B$ be a morphism in $\mathcal{C}_L$ and M be a rank n Fontaine-Laffaille module over A. Then $U(M \bigotimes_A B) = U(M) \bigotimes_A B$.*

Proof. B is finitely generated as an A-module so there exists some presentation of the form $A^r \longrightarrow A^s \longrightarrow B \longrightarrow 0$. Since M is free and hence flat over A , we may tensor this exact sequence with M and then deduce the result from the exactness and additivity of the functor U. □

Proposition 2.2.14. *Let $0 \leq r \leq p-2$ and $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be a residual Fontaine-Laffaille representation with Hodge-Tate weights in the range $[0, r]$. Then the subfunctor $\mathcal{D}_{\overline{\rho}}^{\square,cris} \subseteq \mathcal{D}_{\overline{\rho}}^{\square}$, which associates to $A \in \mathcal{C}_L$ the set of crystalline lifts of $\overline{\rho}$ to A with Hodge-Tate weights in the range $[0, r]$, is a deformation condition in the sense of Definition 1.1.18.*

Remark 2.2.15. This is a straightforward extension of section 2 of [28].

Proof. It suffices to check that the property of a representation being in the image of the Fontaine-Laffaille functor U_S is stable under direct sums and subquotients. Since U_S is additive and an (anti-)equivalence between $MF^{f,r}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ and $\mathrm{Rep}_A^{cris,\leq r}(G_K)$ for any $A \in \mathcal{C}_L$, and since $MF^{f,p-1}_{tor,\mathcal{O}_K}$ is abelian, the result follows. □

Definition 2.2.16. As for characteristic zero representations, given two Fontaine-Laffaille representations ρ_1, ρ_2 valued in a ring $A \in \mathcal{C}_L$ and an embedding σ, we write $HT_\sigma(\rho_1) > HT_\sigma(\rho_2)$ if there is an integer i such that all elements of $HT_\sigma(\rho_1)$ are greater than or equal to i, and all elements of $HT_\sigma(\rho_2)$ are strictly less than i.

Proposition 2.2.17. *Let ρ_i $(i = 1, 2)$ be residually irreducible Fontaine-Laffaille representations valued in a ring $A \in \mathcal{C}_L$. Assume that for each σ, we have that $HT_\sigma(\rho_1) > HT_\sigma(\rho_2)$, and that $\overline{\rho_1} \not\cong \chi_p \bigotimes \overline{\rho_2}$. Then every extension of ρ_2 by ρ_1 valued in A is Fontaine-Laffaille.*

Proof. Suppose $\dim(\rho_i) = n_i$ $(i = 1, 2)$. Then the space of crystalline extensions is of dimension $n_1 n_2 [K : \mathbb{Q}_p]$ by Proposition 2.4.2 and the discussion following it. On the other hand G_K is of cohomological dimension 2, and by the Euler characteristic formula combined with local Tate duality,

$$\chi(G_K, \mathrm{Hom}(\rho_2, \rho_1)) = \frac{h^0(\rho_2, \rho_1) h^0(\rho_1, \chi_p \bigotimes \rho_2)}{h^1(\rho_2, \rho_1)}$$

using the notation that, for representations ρ, ρ' and integer j, $h^j(\rho, \rho')$ is the size of $H^j(G_K, \mathrm{Hom}(\rho, \rho'))$.

The Euler characteristic is $\chi(G_K, \mathrm{Hom}(\rho_2, \rho_1)) = (\# A)^{-n_1 n_2 [K:\mathbb{Q}_p]}$. Since the representations ρ_i are irreducible and distinct, we deduce from Schur's lemma that $h^0(\rho_2, \rho_1) = 0$ and $h^0(\rho_1, \chi_p \bigotimes \rho_2) = 0$. The result follows. $\square$

Remark 2.2.18. Under the conditions of the above proposition except assuming now that $\rho_1 \cong \chi_p \bigotimes \rho_2$, the same calculation gives

$$h^1(\rho_2, \rho_1) = (\# A) h^1_{cris}(\rho_2, \rho_1)$$

where $h^1_{cris}(\rho_2, \rho_1)$ denotes the size of the set of all extensions of ρ_2 by ρ_1 valued in A which correspond to Fontaine-Laffaille representations of G_K.

2.3 Classification of low rank Fontaine-Laffaille modules

It is a straightforward exercise to classify Fontaine-Laffaille modules of low rank. Before we do this we introduce the following notation for Fontaine-Laffaille modules.

Let $M \in MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ of rank n. For each $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, pick an A-basis $(e_{r,\sigma})_{r=1}^n$ for M_σ by repeatedly extending A-bases of the direct summands M_σ^{p-1}, M_σ^{p-2}, $\ldots, M_\sigma^0$, ordered in such a way that each M_σ^j is spanned by $e_{r_j,\sigma}, e_{r_j+1,\sigma}, \ldots, e_{n,\sigma}$ for appropriate integers r_j. We then form a matrix ϕ_σ where the r^{th} column gives the coefficients required to write $\phi^{j(\sigma,r)}(e_{r,\sigma})$ in terms of the $(e_{r,\sigma\cdot Frob^{-1}})_{r=1}^n$. Here $j(\sigma, r)$ is the labelled Hodge-Tate weight for σ associated to $e_{r,\sigma}$; that is, the largest j such that $e_{r_j,\sigma} \in M_\sigma^j$. Note that, under this construction, a necessary and sufficient condition for a filtered $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$-module to be a Fontaine-Laffaille module is that each $\phi_\sigma \in GL_n(A)$, since the ϕ-action on the whole of M is then specified by the rules $\phi^i_M|_{M^{i+1}} = p\phi^{i+1}_M$ for all i.

We now proceed with the classification.

Example 2.3.1. Let M be a rank 1 Fontaine-Laffaille module over A, with labelled Hodge-Tate weights $(i_\sigma)_\sigma$. For each σ, the map

$$\phi_\sigma^{i_\sigma} : M_\sigma^{i_\sigma} \longrightarrow M_{\sigma\cdot Frob^{-1}}$$

is an isomorphism, so specified by a unit $\phi_\sigma^{i_\sigma} \in A^\times$. Two such modules are isomorphic if there is an $x \in (\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A)^\times$ mapping one $(\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A)$-basis to another; in other words, the collection $(\phi_\sigma^{i_\sigma})_\sigma$ is equivalent to the collection

$$(\phi_\sigma^{i_\sigma}(\frac{\tau(x)}{x})_{\sigma\cdot Frob^{-1}})_\sigma = (\phi_\sigma^{i_\sigma}\frac{x_\sigma}{x_{\sigma\cdot Frob^{-1}}})_\sigma.$$

It is straightforward to see that $(\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A)^\times$ modulo this relation is isomorphic to $A^\times$ by sending $(\phi_\sigma^{i_\sigma})_\sigma$ to $\prod_\sigma \phi_\sigma^{i_\sigma}$. Thus a Fontaine-Laffaille rank one character of G_K on A is specified by the collection $(i_\sigma)_\sigma$ of labelled Hodge-Tate weights together with a unit $\phi \in A^\times$. In particular we see that the rank 1 Fontaine-Laffaille deformation functor over an unramified base is smooth of relative dimension 1. The reader should compare this with remark 3.1.2.

Remark 2.3.2. In the situation where $K = \mathbb{Q}_p$ and recalling the notation of example 1.2.4, we may take $\omega_1 = \chi_p$, and conclude from Proposition 1.2.5 that every character $\chi : G_K \longrightarrow k_L^\times$ is of the form χ_p^h for some h in the range $1 \le h \le p-2$, up to unramified twist. In particular, we see that all rank 1 mod p representations of $G_{\mathbb{Q}_p}$ are Fontaine-Laffaille (with h being the Hodge-Tate weight).

Example 2.3.3. Suppose $K = \mathbb{Q}_p$. We classify the rank 2 Fontaine-Laffaille modules over A.

Let $i \le j$ be the Hodge-Tate weights of a rank 2 Fontaine-Laffaille module M. If $i = j$ then ϕ^i is specified up to a choice of basis by a matrix $\phi^i \in GL_2(A)$. The corresponding representation will be the i^{th} cyclotomic twist of an unramified representation whose action on Frobenius is specified by ϕ^i.

So suppose $i < j$, and consider the A-line $M^j \subset M$. For convenience, we use the notation $N \bigoplus_f M$ for extensions of Fontaine-Laffaille modules, details of which can be found in the proof of Proposition 2.4.2. Also, for an integer i in the range

$0 \leq i \leq p-2$ and a unit $a \in A^\times$, (i,a) will denote the rank 1 Fontaine-Laffaille module of weight i and parameter a as in the previous example. There are then 3 cases.

1. $\phi^j(M^j) = M^j$. In this case, there exist units $a, d \in A^\times$ depending only on M such that M splits as $(i,a) \bigoplus (j,d)$ (in other words, $M \cong (i,a) \bigoplus_0 (j,d)$ in the notation of Proposition 2.4.2).

2. $\phi^j(M^j) \bigoplus M^j = M$. In this case M is irreducible and is specified by 2 parameters $a \in \mathfrak{m}_A$, $c \in A^\times$ which depend only on M. A basis for M as a filtered module can be chosen such that the matrix of ϕ takes the form

$$\begin{pmatrix} a & 1 \\ c & 0 \end{pmatrix}.$$

   This follows from the observation that any basis $\{e\}$ for M^j gives a basis $\{\phi^j(e), e\}$ for M. We require that $a \in \mathfrak{m}_A$ since otherwise the module constructed here is isomorphic to the non-split extension $(i, a+y\cdot p^{j-i}) \bigoplus_{-y^{-1}} (j, -y)$ in the notation of Proposition 2.4.2, where $y \in A^\times$ is any root of the polynomial $p^{j-i}y^2 + ay - c$ (that this polynomial has a root in $A^\times$ follows from an application of Hensel's lemma).

   The corresponding representation is irreducible. Note that in particular, when $A = \mathbb{F}_p$, there is only one such representation up to unramified twist. One sees also that if $\overline{\rho}$ corresponds to such a Fontaine-Laffaille module, then the crystalline deformation problem for $\overline{\rho}$ is smooth of relative dimension 2 over $\mathbb{Z}_p$, and so (since $\overline{\rho}$ is irreducible of dimension 2), the crystalline framed deformation problem is smooth of relative dimension 5. The reader should compare this with Theorem 3.1.1.

3. $\phi^j(M^j) \neq M^j$ but $\phi^j(M^j) \cap M^j \neq 0$. In this case, by Hensel's lemma, one shows that $\exists 0 \neq b \in \mathfrak{m}_A$ (determined by M up to unit), and $a, d \in A^\times$ (determined uniquely by M) such that M is isomorphic to the non-split extension $(i, a) \bigoplus_b (j, d)$ in the notation of Proposition 2.4.2. Explicitly, extend a basis $\{f\}$ of M^j to a basis $\{e, f\}$ of M and suppose the matrix of ϕ takes the form

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}.$$

By assumption, $b \in \mathfrak{m}_A$ and $a, d \in A^\times$. By Hensel's lemma, there is some $y \in A$ with $c + dyp^{j-i} = ay + by^2p^{j-i}$; putting $e' = e + yf$ gives the matrix of ϕ the required upper triangular form.

All such modules, and the corresponding representations, are residually split.

In particular in the case when $\overline{\rho}$ is upper triangular with strictly decreasing Hodge-Tate weights, the upper triangular crystalline framed deformation problem associated to $\overline{\rho}$ (as in Theorem 3.1.5) is smooth of relative dimension 3, since one must specify lifts of the quantities $\overline{a}$, $\overline{b}$, and $\overline{d}$. The reader should compare this with Corollary 3.1.6.

Remark 2.3.4. In the situation where $K = \mathbb{Q}_p$, we know from Proposition 1.2.5 that every absolutely irreducible representation $\overline{\rho} : G_K \longrightarrow GL_2(k_L)$ is of the form $\mathrm{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p}(\omega_2^h)$ up to unramified twist, for some h which is not divisible by $p+1$ and lies in the range $1 \leq h \leq p^2 - 2$ (here $\mathbb{Q}_{p^2}$ denotes the unique quadratic unramified extension of $\mathbb{Q}_p$). Note that the rank 2 Fontaine-Laffaille modules with equal Hodge-Tate weights classified above do not correspond to absolutely irreducible representations. Writing $h = ip + j$ for $0 \leq i < j \leq p$ and $i \leq p-2$, we then have $\overline{\rho} = \chi_p^i \mathrm{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p}(\omega_2^{j-i})$ up to unramified twist.

1. If $j \leq p-2$ then we have constructed the Fontaine-Laffaille representation above (with (i, j) being the Hodge-Tate weights).

2. If $j = p-1$, Fontaine-Laffaille theory can be extended to this situation since U_S is fully faithful on the full subcategory $MF_{tor,\mathcal{O}_K}^{f,p-1}{}''$ (part 2 of Theorem 2.2.8).

3. Finally, if $j = p$, then observing that ω_2 and ω_2^p are conjugate by $\mathrm{Gal}(\mathbb{Q}_{p^2}/\mathbb{Q}_p)$ (and so have the same induction to $\mathbb{Q}_p$), we have $\overline{\rho} = \mathrm{Ind}_{\mathbb{Q}_{p^2}}^{\mathbb{Q}_p}(\omega_2^{i+1})$, the Fontaine-Laffaille representation with Hodge-Tate weights 0 and $i+1$ (note that we necessarily have that $i+1 \leq p-1$).

In particular, we see that all rank 2 absolutely irreducible mod p representations of $G_{\mathbb{Q}_p}$ are Fontaine-Laffaille (after extending the definition to include case 2 above).

2.4 Extensions of Fontaine-Laffaille modules

Finally we establish an important result on the structure of the group of extension classes in the category of Fontaine-Laffaille modules. We will need the following lemma.

Lemma 2.4.1. *Let $A \in \mathcal{C}_L$. Let M and N be Fontaine-Laffaille modules over A, and $y \in Hom_{Fil,\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M, N)$. Suppose we have elements $m^i \in M^i$ such that $\sum_i \phi_M^i(m^i) = 0$. Then $\sum_i \phi_N^i(y(m^i)) = 0$.*

Proof. Repeatedly extend bases from the direct summands M^i until we obtain a basis for M. Denoting the span of the basis elements introduced when extending to M^j as $\tilde{M}^j$, we write each m^i as a sum:

$$m^i = \sum_{j \geq i} m^{i,j}$$

where $m^{i,j} \in \tilde{M}^j$. Observe that ϕ^j_M is injective on $\tilde{M}^j$ and $M = \bigoplus_j \phi^j_M(\tilde{M}^j)$; also, since y preserves the filtration, $y(m^{i,j}) \in N^j$. By assumption,

$$0 = \sum_i \phi^i_M(m^i) = \sum_{i \leq j} \phi^j_M(p^{j-i} m^{i,j})$$

which implies $\sum_{i=1}^j p^{j-i} m^{i,j} = 0$ for all j. Thus

$$\sum_i \phi^i_N(y(m^i)) = \sum_{i \leq j} p^{j-i} \phi^j_N(y(m^{i,j})) = 0$$

as required. □

We can now state the main result of this section.

Proposition 2.4.2. *Suppose $A \in \mathcal{C}_L$. Let M and N be Fontaine-Laffaille modules over A. Then we have an exact sequence:*

$$0 \longrightarrow Hom_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M, N) \longrightarrow Hom_{Fil,\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M, N) \longrightarrow$$
$$\longrightarrow Hom_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M, N) \longrightarrow Ext^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M, N) \longrightarrow 0.$$

Proof. The construction of this sequence is similar to that performed in the proof of Proposition 2.16 in [14]. For convenience we give the details here.

Given $y \in \mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$, define $\phi(y) \in \mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$ as follows: if $m = \sum_i \phi^i_M(m^i)$ then $\phi(y)(m) = \sum_i \phi^i_N(y(m^i))$. This is well defined by Lemma 2.4.1. It is then straightforward to see that $\mathrm{Ker}(\phi - 1) = \mathrm{Hom}_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$.

For any $f \in \mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$ we can construct an extension which we will denote $N \bigoplus_f M$ as follows: the filtration structure is $(N \bigoplus_f M)^i = N^i \bigoplus M^i$, and ϕ^i sends the pair (n^i, m^i) to $(\phi^i_N(n^i) + f(\phi^i_M(m^i)), \phi^i_M(M^i))$. It is trivial to check that the resulting module satisfies the definition. In this way we have obtained a map $\mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N) \longrightarrow \mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$. Unraveling, we see that a map f lies in the kernel precisely when there is a θ, necessarily of the form $\theta(n,m) = (n + y(m), m)$ with $y \in \mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$, fitting in to the commutative diagram

$$\begin{array}{ccccccccc}
0 & \longrightarrow & N & \longrightarrow & N \bigoplus_f M & \longrightarrow & M & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & N & \longrightarrow & N \bigoplus M & \longrightarrow & M & \longrightarrow & 0
\end{array}$$

(where the leftmost and rightmost arrows are the identity) and such that $\phi^i_N(y(m^i)) = y(\phi^i_M(m^i)) + f(\phi^i_M(m^i))$ for all i and elements $m^i \in M^i$, by ϕ-compatibility of θ. In other words, the kernel consists precisely of those f of the form $\phi(y) - y$ for $y \in \mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M,N)$.

It remains to prove that this map is surjective. Given $D \in \mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M,N)$, we have $D \cong N \bigoplus M$ as filtered modules. Pick filtration compatible linear sections $a : D \longrightarrow N$ and $b : M \longrightarrow D$. We may then define a map $f : M \longrightarrow N$ by sending $\phi^i_M(m^i)$ to $a(\phi^i_D b(m^i))$ and extending linearly. Again this is well-defined by Lemma 2.4.1, thus exhibiting D as lying in same class as $N \bigoplus_f M$. The result follows. $\square$

Note that the formation of this exact sequence commutes with base extension in the sense that, if $A \longrightarrow B$ is a map in the category $\mathcal{C}_L$, the resulting exact sequence over B is obtained by tensoring each term with B over A. Also by Fontaine-Laffaille theory, $\mathrm{Hom}_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M, N) \cong \mathrm{Hom}_{A[G_K]}(U_S(N), U_S(M))$ provided that M and N lie in $MF^{f,r}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ (by Theorem 2.2.8, part 3). In particular, the first term in the exact sequence is free over A. The third term is also free over $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$ and hence over A. Thus by a repeated application of Lemma 2.5.2 below, we see that all terms are in fact free over A.

Definition 2.4.3. We denote the A-rank of $\mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M, N)$ as $d_{M,N}$, or simply as d_M in the special case where $M = N$. Observe that this quantity depends only on the labelled Hodge-Tate weight structure of M and N.

There are two notable cases.

1. Suppose for every $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$ that $HT_\sigma(M) > HT_\sigma(N)$. There is thus an integer i (depending on σ) such that $N^i_\sigma = 0$ and $M^i_\sigma = M_\sigma$. It follows that $\mathrm{Hom}_{\mathrm{Fil},\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M, N) = 0$, and so $d_{M,N} = 0$ and $\mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M, N)$ has rank $[K : \mathbb{Q}_p]\mathrm{rk}(M_1)\mathrm{rk}(M_2)$ over A by Proposition 2.4.2. Note that this is the maximum possible value for $\mathrm{rk}_A(\mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M, N))$. We call this case the *Hodge-Tate* case.

2. Suppose conversely that for every σ, $HT_\sigma(N) > HT_\sigma(M)$. There is thus an integer i (depending on σ) such that $N^i_\sigma = N_\sigma$ and $M^i_\sigma = 0$. Pick any $f \in \mathrm{Hom}_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M, N)$ and $m \in M_{\sigma \cdot Frob^{-1}}$. We may write $m = \sum_{j<i} \phi^j_M(m^j)$ for appropriate $m^j \in M^j_\sigma$. We then have

$$f(m) = \sum_{j<i} \phi_N^j(f(m^j)) = \sum_{j<i} p^{i-j}\phi_N^i(f(m^j)) \in p \cdot N_{\sigma \cdot Frob^{-1}}$$

and so $f(M) \subseteq p \cdot N$. By induction one then sees that in fact $f(M) \subseteq p^r \cdot N$ for every $r \in \mathbb{N}$, and so since $p^r \cdot N = 0$ for sufficiently large r we conclude that $f = 0$.

On the other hand, since $N_\sigma^i = N_\sigma$ and $M_\sigma^i = 0$, it follows that every element of $\mathrm{Hom}_{\mathcal{O}_K \otimes_{\mathbb{Z}_p} A}(M, N)$ preserves the filtration. We conclude that $d_{M,N} = [K : \mathbb{Q}_p]\mathrm{rk}(M_1)\mathrm{rk}(M_2)$ and $\mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M, N) = 0$. We call this case the *anti-Hodge-Tate* case.

Remark 2.4.4. Comparing the Hodge-Tate case to standard results on extensions of crystalline representations gives a justification for the sign convention mentioned in Remark 2.2.7. Namely, given $M, N \in MF^{f,p-1}_{tor,\mathcal{O}_K} \bigotimes_{\mathbb{Z}_p} A$ such that, for all σ, the labelled Hodge-Tate weights for M are bigger than all of those for N, we have that any extension of $U_S(N)$ by $U_S(M)$ is Fontiane-Laffaille by the given inequality on Hodge-Tate weights (using the convention that the cyclotomic character has weight $+1$). This corresponds to the Hodge-Tate case above (recalling that U_S is contravariant), and so we get the equality of Remark 2.2.7.

2.5 Commutative algebra

We end this chapter with a few simple commutative algebra facts that were referred to in this chapter or will be needed in the following.

We record the following lemma, whose proof is straightforward.

Lemma 2.5.1. *Let $A \in \mathcal{C}_L$. Then the map $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A \longrightarrow \prod_\sigma A$ which sends $x \otimes y$ to $(\sigma(x) \cdot y)_\sigma$, where the product runs over all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, is an isomorphism.*

Given an element of $\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} A$, we may thus refer to its σ-component for any embedding σ.

Lemma 2.5.2. *Let A be a local ring.*

1. *Suppose N is a free finitely generated A-module and M is a free submodule of N such that the reduced map $\overline{M} \longrightarrow \overline{N}$ is an injection. Then M is a direct summand.*

2. *Let*

$$0 \longrightarrow M \longrightarrow D \longrightarrow N \longrightarrow 0$$

be an exact sequence of finitely generated A-modules with M and N free over A. Then D is free.

Proof. 1. Choose a basis for $\overline{N}$ which contains a basis for $\overline{M}$. By Nakayama's lemma we may lift this to a generating set for N containing a generating set for M. Since N is free, the elements of this set not in the generating set for M generate a complement to M in N.

2. N is free over A, so projective. The exact sequence therefore splits over A, giving the result. □

Chapter 3

Smoothness of crystalline framed deformation functors and universally twistable lifts

In this chapter we prove the main results of this thesis on the representability and formal smoothness of framed deformation functors associated to various classes of Fontaine-Laffaille Galois representations. We also provide calculations on the dimensions of these functors, and apply the main theorems to the problem of the existence of universally twistable lifts of Fontaine-Laffaille representations.

3.1 Representability and formal smoothness

The first main result we demonstrate is the smooth representability of the framed deformation problem associated to an irreducible Fontaine-Laffaille representation. Note that in [10], Corollary 2.4.3, a similar result is demonstrated for (not necessarily irreducible) crystalline representations with the property that, for all $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, the labelled Hodge-weights for σ occur with multiplicity 1 (so in particular, $n \leq p+1$). However we cannot deduce theorem 3.1.1 below from this. For example, when $K = \mathbb{Q}_p$

and $n = 3$, the Fontaine-Laffaille module with weights $\{0,0,1\}$ and ϕ-action specified by the matrix

$$\begin{pmatrix} 0 & 0 & 1 \\ 1 & 0 & 0 \\ 1 & 1 & 0 \end{pmatrix}$$

can be shown by a simple calculation to be irreducible, even though the Hodge-Tate weight 0 occurs with multiplicity 2. So instead we are forced to take a different approach, motivated by a counting argument from [28] which there is applied to flat deformations.

Theorem 3.1.1. *Let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be an irreducible Fontaine-Laffaille representation, with associated crystalline framed deformation functor $\mathcal{D}_{\overline{\rho}}^{\square,cris}$. Let $\overline{M}$ be the rank n Fontaine-Laffaille module associated with $\overline{\rho}$. Then $\mathcal{D}_{\overline{\rho}}^{\square,cris}$ is represented by a power series ring over $\mathcal{O}_L$ in $n^2([K : \mathbb{Q}_p] + 1) - d_{\overline{M}}$ variables.*

Proof. The argument is based on a generalisation of that found in [28].

By Proposition 2.4.2 we have (since $\overline{\rho}$ is irreducible) that $\mathrm{Ext}^1_{MF}(\overline{M}, \overline{M})$ is of rank $n^2[K : \mathbb{Q}_p] + 1 - d_{\overline{M}}$ over k_L. From the theory of representable functors (in particular, corollary 1.1.11), we see that the associated deformation problem is representable by a ring $R_{\overline{M}}$, and that there is a surjection

$$\mathcal{O}_L[[(T_i)_{i=1}^{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}}]] \twoheadrightarrow R_{\overline{M}}$$

of $\mathcal{O}_L$-algebras. It suffices to prove that this is in fact an isomorphism, since in this case the associated framed deformation problem will be represented by a power series ring over $\mathcal{O}_L$ in $\mathrm{rk}_{\mathcal{O}_L}(R_{\overline{M}}) + n^2 - 1$ variables (as $\overline{\rho}$ is irreducible).

Supposing without loss of generality that L is unramified over $\mathbb{Q}_p$, we count for each $r \in \mathbb{N}$ the lifts M of $\overline{M}$ to $\mathcal{O}_L/p^r$. We will show that there are precisely $q^{(r-1)(n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}})}$, where q is the size of the residue field of L; since this is the number of $\mathcal{O}_L/p^r$-points of $\mathcal{O}_L[[(T_i)_{i=1}^{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}}]]$ we deduce that the above map is an isomorphism (for any f in the kernel, $f(t_1, t_2, \ldots, t_{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}}) = 0$ whenever $t_1, t_2, \ldots, t_{n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}}} \in p\mathcal{O}_L$, which implies $f = 0$).

We are thus seeking matrices $\phi \in GL_n(\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} \mathcal{O}_L/p^r)$ with specified reduction $\overline{\phi} \in GL_n(\mathcal{O}_K \bigotimes_{\mathbb{Z}_p} k_L)$, determined up to τ-semilinear conjugation by a matrix R which preserves the filtration on M and reduces to the identity modulo p.

There are $q^{(r-1)n^2[K:\mathbb{Q}_p]}$ choices for ϕ, and $q^{(r-1)d_{\overline{M}}}$ choices for R. Of these, since M is irreducible, q^{r-1} commute with each ϕ in the sense that τ-semilinear conjugation by R preserves ϕ; or in other words, that

$$\phi_\sigma R_\sigma = R_{\sigma \cdot Frob^{-1}} \phi_\sigma$$

for every $\sigma : K \hookrightarrow L$. There are thus $q^{(r-1)(n^2[K:\mathbb{Q}_p]+1-d_{\overline{M}})}$ lifts of $\overline{M}$ up to isomorphism, as required. $\square$

Remark 3.1.2. If $n = 1$ then $d_{\overline{M}} = [K : \mathbb{Q}_p]$ and so $\mathcal{D}_{\overline{\rho}}^{\square, cris}$ is smooth of rank 1, as expected.

We will now extend this result to the situation where the representation consists of a number of diagonal blocks. To this end, we make the following definition.

Definition 3.1.3. Suppose

$$\rho = \begin{pmatrix} \rho_1 & * & \dots & * \\ 0 & \rho_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_r \end{pmatrix}$$

is a block upper triangular representation of a group G valued in some ring A. Given a positive integer $i \leq r$, the i^{th} *truncation* of ρ is the representation

$$\rho_{\leq i} = \begin{pmatrix} \rho_1 & * & \dots & * \\ 0 & \rho_2 & \dots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \dots & \rho_i \end{pmatrix}.$$

Remark 3.1.4. Assuming that ρ and $\rho_1, \rho_2, \dots, \rho_r$ as above are Fontaine-Laffaille representations of G_K valued in some ring $A \in \mathcal{C}_L$, all truncations $\rho_{\leq i}$ of ρ are also Fontaine-Laffaille. This follows from Theorem 2.2.8 together with the fact that $MF^{f,p-1}_{tor,\mathcal{O}_K}$ is an abelian category ([19], Proposition 1.8).

In the situation of the above remark, we will denote the Fontaine-Laffaille module associated to $\rho_{\leq i}$ as $M_{\leq i}$, and the Fontaine-Laffaille module associated to $\rho_{\leq i-1}$ as $M_{<i}$, for $i \leq r$. Note that $M_{\leq i} \in \mathrm{Ext}^1_{MF^{f,p-1}_{tor,\mathcal{O}_K} \otimes_{\mathbb{Z}_p} A}(M_{<i}, M_i)$, where M_i is the Fontaine-Laffaille module associated to ρ_i.

We are now in a position to prove the following theorem.

Theorem 3.1.5. *For $i = 1, 2, \dots, r$, let $\overline{\rho_i} : G_K \longrightarrow GL_{n_i}(k_L)$ be irreducible Fontaine-Laffaille representations, with associated framed deformation functors $\mathcal{D}_i^{\square,cris}$, and* *associated Fontaine-Laffaille modules* $\overline{M_i}$*, of rank* n_i*. Fix a Fontaine-Laffaille representation* $\overline{\rho}$ *which is block upper triangular with* $\overline{\rho_1}, \overline{\rho_2}, \ldots, \overline{\rho_r}$ *on the diagonal, and define a functor* $\mathcal{F}_{\overline{\rho}} : \mathcal{C}_L \longrightarrow Set$ *which sends a ring* A *to the set of crystalline lifts* ρ *of* $\overline{\rho}$ *to* A *of the form*

$$\rho = \begin{pmatrix} \rho_1 & * & \ldots & * \\ 0 & \rho_2 & \ldots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \ldots & \rho_r \end{pmatrix}$$

for $\rho_i \in \mathcal{D}_i^{\square,cris}(A)$ *(*$i = 1, 2, \ldots, r$*). Then* $\mathcal{F}_{\overline{\rho}}$ *is representable. Moreover, the natural map* $\mathcal{F}_{\overline{\rho}} \longrightarrow \prod_{i=1}^r \mathcal{D}_i^{\square,cris}$ *is smooth of relative dimension* $\sum_{i=1}^r d_i$*, where*

$$d_i = (\sum_{j<i} n_j) n_i([K : \mathbb{Q}_p] + 1) - d_{\overline{M_{<i}}, \overline{M_i}}.$$

Proof. We prove this result by induction on r.

For $i = 1, 2, \ldots, r$, let $\mathcal{F}_{\overline{\rho},i}$ be the crystalline lift functor corresponding to the i^{th} truncation of $\overline{\rho}$.

We first show that $\mathcal{F}_{\overline{\rho},2} \longrightarrow \mathcal{D}_1^{\square,cris} \times \mathcal{D}_2^{\square,cris}$ is relatively representable and smooth of relative dimension d_2. To do this, it suffices to prove that the collection $Z^1_{crys,A}(\rho_2, \rho_1)$ of crystalline cocycles is a free A-module of rank $n_1 n_2([K : \mathbb{Q}_p] + 1) - d_{\overline{M_1}, \overline{M_2}}$, for any $A \in \mathcal{C}_L$ and pair of lifts $(\rho_1, \rho_2) \in (\mathcal{D}_1^{\square,cris} \times \mathcal{D}_2^{\square,cris})(A)$. Letting $B^1_{crys,A}(\rho_2, \rho_1)$ denote the collection of crystalline coboundaries, we have an exact sequence

$$0 \longrightarrow \mathrm{Hom}_{G_K}(\rho_2, \rho_1) \longrightarrow Mat_{n_1 \times n_2}(A) \longrightarrow B^1_{crys,A}(\rho_2, \rho_1) \longrightarrow 0$$

and so $B^1_{crys,A}(\rho_2, \rho_1)$ is free over A of rank $n_1 n_2 - \dim(\mathrm{Hom}_{G_K}(\rho_2, \rho_1))$ by Lemma 2.5.2

(part 1).

By Proposition 2.4.2 and the discussion following it, $H^1_{crys,A}(\rho_2, \rho_1)$ is free over A of rank $n_1 n_2[K : \mathbb{Q}_p] + \dim(\mathrm{Hom}_{G_K}(\rho_2, \rho_1)) - d_{\overline{M_1}, \overline{M_2}}$. From the exact sequence

$$0 \longrightarrow B^1_{crys,A}(\rho_2, \rho_1) \longrightarrow Z^1_{crys,A}(\rho_2, \rho_1) \longrightarrow H^1_{crys,A}(\rho_2, \rho_1) \longrightarrow 0$$

and Lemma 2.5.2 (part 2), we deduce the result for $r = 2$.

Now suppose that $\mathcal{F}_{\overline{\rho},k} \longrightarrow \mathcal{F}_{\overline{\rho},k-1} \times \mathcal{D}_k^{\square,cris}$ is relatively representable and smooth of relative dimension d_k. As before, to show that $\mathcal{F}_{\overline{\rho},k+1} \longrightarrow \mathcal{F}_{\overline{\rho},k} \times \mathcal{D}_{k+1}^{\square,cris}$ is relatively representable and smooth of relative dimension d_{k+1}, it suffices to prove that $Z^1_{crys,A}(\rho_{k+1}, \rho_{\leq k})$ is free over A of rank d_{k+1} for any $A \in \mathcal{C}_L$ and lifts $\rho_{\leq k}, \rho_{k+1}$. But the exact sequences

$$0 \longrightarrow \mathrm{Hom}_{G_K}(\rho_{k+1}, \rho_{\leq k}) \longrightarrow Mat_{\sum_{i=1}^k n_i \times n_{k+1}}(A) \longrightarrow B^1_{crys,A}(\rho_{k+1}, \rho_{\leq k}) \longrightarrow 0$$

and

$$0 \longrightarrow B^1_{crys,A}(\rho_{k+1}, \rho_{\leq k}) \longrightarrow Z^1_{crys,A}(\rho_{k+1}, \rho_{\leq k}) \longrightarrow H^1_{crys,A}(\rho_{k+1}, \rho_{\leq k}) \longrightarrow 0,$$

combined with Proposition 2.4.2, the discussion following it, and Lemma 2.5.2, imply that $Z^1_{crys,A}(\rho_{k+1}, \rho_{\leq k})$ is free over A of rank $([K : \mathbb{Q}_p] + 1)n_{k+1} \sum_{i=1}^k n_i - d_{\overline{M_{\leq k}}, \overline{M_{k+1}}}$, which is precisely d_{k+1}. We conclude the result by induction. □

Corollary 3.1.6. *Let notation be as in Theorem 3.1.5. Then $\mathcal{F}_{\overline{\rho}}$ is represented by a power series ring over $\mathcal{O}_L$ in $([K : \mathbb{Q}_p] + 1)(\sum_{i,j:i \leq j} n_i n_j) - \sum_{i=1}^r d_{\overline{M_{\leq i}}, \overline{M_i}}$ variables.*

Proof. By Theorem 3.1.1, each $\mathcal{D}_i^{\square,cris}$ for $i = 1, 2, \ldots, r$ is smooth of relative dimension c_i, for $c_i = n_i^2([K : \mathbb{Q}_p] + 1) - d_{\overline{M_i}}$. The result then follows from Theorem 3.1.5 after observing that

$$c_i + d_i = (\sum_{j \leq i} n_j) n_i([K : \mathbb{Q}_p] + 1) - d_{\overline{M_{\leq i}}, \overline{M_i}},$$

using the fact that $\overline{M_{\leq i}} = \overline{M_{<i}} \bigoplus \overline{M_i}$ as filtered modules. □

Remark 3.1.7. In Corollary 2.4.3 of [10] it is shown that if $\overline{\rho}$ is an n-dimensional crystalline representation of G_K with labelled Hodge-Tate weights all of multiplicity 1, then the crystalline framed deformation problem for $\overline{\rho}$ is representable by a power series ring over $\mathcal{O}_L$ in $n^2 + [K : \mathbb{Q}_p]\frac{n(n-1)}{2}$ variables. Note that if $\overline{\rho}$ is irreducible and corresponds to the Fontaine-Laffaille module $\overline{M}$ then $d_{\overline{M}} = [K : \mathbb{Q}_p]\frac{n(n+1)}{2}$ and so this result is a special case of Theorem 3.1.1.

Example 3.1.8. Suppose $\overline{\rho}$ is an n-dimensional upper triangular Fontaine-Laffaille representation with Fontaine-Laffaille characters $\overline{\chi_1}, \overline{\chi_2}, \ldots, \overline{\chi_n}$ on the diagonal, and that $\overline{\chi_j}$ has labelled Hodge-Tate weights $i_{j,\sigma}$ for $1 \leq j \leq r$. Then $\mathcal{F}_{\overline{\rho}}$ is represented by a power series ring over $\mathcal{O}_L$ in

$$\frac{n(n+1)}{2} + \sum_{a \leq b} \#\{\sigma \mid i_{b,\sigma} < i_{a,\sigma}\}$$

variables. In particular, if $\overline{\rho}$ is ordinary in the sense that $i_{b,\sigma} < i_{a,\sigma}$ for all σ when $a < b$ then the dimension is $n + \frac{n(n-1)}{2}([K : \mathbb{Q}_p] + 1)$.

Corollary 3.1.9. *Let $\overline{\rho}$ be an n-dimensional upper triangular Fontaine-Laffaille representation with Fontaine-Laffaille characters $\overline{\chi_1}, \overline{\chi_2}, \ldots, \overline{\chi_n}$ on the diagonal. Suppose $\overline{\rho}$ is ordinary in the sense that all labelled Hodge-Tate weights of the characters $\overline{\chi_j}$*

strictly decrease down the diagonal, and the characters $\overline{\chi_1}, \overline{\chi_2}, \ldots, \overline{\chi_n}$ *are (residually) pairwise distinct. Then all crystalline deformations of* $\overline{\rho}$ *are ordinary.*

Proof. From Corollary 2.4.3 of [10] and the calculation in the above example, we see that

$$\dim_{\mathcal{O}_L} \mathcal{D}_{\overline{\rho}}^{\square,cris} - \dim_{\mathcal{O}_L} \mathcal{F}_{\overline{\rho}} = \frac{n(n-1)}{2}$$

where $\mathcal{D}_{\overline{\rho}}^{\square,cris}$ is the functor giving the set of all crystalline lifts of $\overline{\rho}$, and $\mathcal{F}_{\overline{\rho}}$ is as in Theorem 3.1.5. Since the diagonal characters are residually distinct, conjugation by lower triangular unipotent matrices which reduce to the identity acts simply transitively on the set of lifts of $\overline{\rho}$ and each class contains a representative from the image of $\mathcal{F}_{\overline{\rho}}$. Since $\frac{n(n-1)}{2}$ is the dimension of the set of lower triangular unipotent matrices which reduce to the identity, the result follows. $\square$

3.2 Dimension bounds

Suppose $\overline{\rho}$ is irreducible Fontaine-Laffaille of dimension n, and corresponds to the Fontaine-Laffaille module $\overline{M}$. It is straightforward to show that

$$[K : \mathbb{Q}_p] \frac{n(n+1)}{2} \leq d_{\overline{M}} \leq [K : \mathbb{Q}_p] n^2$$

where the quantity $d_{\overline{M}}$ is as defined in definition 2.4.3. Hence the crystalline framed deformation problem for $\overline{\rho}$ is smooth of relative dimension $c_{\overline{\rho}}$, where

$$n^2 \leq c_{\overline{\rho}} \leq n^2 + [K : \mathbb{Q}_p] \frac{n(n-1)}{2}.$$

The lower bound here comes from being able to twist by any representation which is unramified and residually trivial, and is attained only if all labelled Hodge-Tate weights occur with multiplicity n; in other words, when $\overline{\rho}$ is a twist by some character of an unramified representation.

Now suppose $\overline{\rho_1}, \overline{\rho_2}, \ldots, \overline{\rho_r}$ are irreducible Fontaine-Laffaille of dimension n, and correspond to the Fontaine-Laffaille modules $\overline{M_1}, \overline{M_2}, \ldots, \overline{M_r}$. Let $\overline{\rho}$ be a Fontaine-Laffaille representation of G_K which is upper triangular with $\overline{\rho_1}, \overline{\rho_2}, \ldots, \overline{\rho_r}$ on the diagonal. Using the notation of Theorem 3.1.5,

$$(i-1)n^2 \leq d_i \leq ([K:\mathbb{Q}_p]+1)(i-1)n^2$$

for each i, which implies by taking the sum over i that

$$\frac{r(r+1)}{2}n^2 \leq \dim_{\mathcal{O}_L} \mathcal{F}_{\overline{\rho}} \leq \frac{r(r+1)}{2}n^2 + \frac{rn(rn-1)}{2}[K:\mathbb{Q}_p].$$

3.3 Application to universal twistable lifts

We now discuss how the results of this chapter can be applied to the work on so-called "universally twistable lifts" carried out in [20]. We first review some of the definitions in that paper. We temporarily remove all restrictions on notation that have been present up until now, and assume only that K is a finite extension of $\mathbb{Q}_p$, and L is a subfield of $\overline{\mathbb{Q}_p}$ containing the images of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$. Let k_L denote the residue field of L.

Let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be any representation, and denote by $\overline{V}$ the underlying $k_L[G_K]$-module. Suppose $0 = \overline{U_0} \subsetneq \overline{U_1} \subsetneq \ldots \subsetneq \overline{U_r} = \overline{V}$ is an increasing filtration of

$\overline{V}$ by $k_L[G_K]$-submodules, and put $\overline{V_i} = \overline{U_i}/\overline{U_{i-1}}$ for $i = 1, 2, \ldots, r$.

Definition 3.3.1. $\overline{\rho}$ *admits a universally twistable lift with respect to the filtration* $(\overline{U_i})_{i=0}^r$ if there exists lifts V_i of $\overline{V_i}$ to $\mathcal{O}_L$ for $i = 1, 2, \ldots, r$, together with, for every r-tuple of unramified residually trivial characters $(\psi_1, \psi_2, \ldots, \psi_r)$ of G_K, a lift $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$ to $\mathcal{O}_L$, satisfying the following additional properties for $i = 1, 2, \ldots, r$:

1. $V(\psi_1, \psi_2, \ldots, \psi_r)$ has an increasing filtration by $\mathcal{O}_L[G_K]$-submodules $0 = U(\psi_1, \psi_2, \ldots, \psi_r)_0 \subsetneq U(\psi_1, \psi_2, \ldots, \psi_r)_1 \subsetneq \ldots \subsetneq U(\psi_1, \psi_2, \ldots, \psi_r)_r$ $= V(\psi_1, \psi_2, \ldots, \psi_r)$ which are free $\mathcal{O}_L$-direct summands, such that $U(\psi_1, \psi_2, \ldots, \psi_r)_i / U(\psi_1, \psi_2, \ldots, \psi_r)_{i-1} \cong V_i \bigotimes_{\mathcal{O}_L} \psi_i$.

2. The reduction $V(\psi_1, \psi_2, \ldots, \psi_r) \bigotimes_{\mathcal{O}_L} k_L \cong \overline{V}$ induces reductions $U(\psi_1, \psi_2, \ldots, \psi_r)_i \bigotimes_{\mathcal{O}_L} k_L \cong \overline{U_i}$.

3. The submodule $U(\psi_1, \psi_2, \ldots, \psi_r)_i$ depends up to isomorphism only on $(\psi_1, \psi_2, \ldots, \psi_i)$, and not on $(\psi_{i+1}, \psi_{i+2}, \ldots, \psi_r)$.

Definition 3.3.2. $\overline{\rho}$ *admits a universally twistable lift* if it does so with respect to a saturated filtration $(\overline{U_i})_{i=0}^r$; here we say that a filtration $(\overline{U_i})_{i=0}^r$ is *saturated* if the graded pieces $\overline{V_i} = \overline{U_i}/\overline{U_{i-1}}$ are absolutely irreducible, for all $i = 1, 2, \ldots, r$.

Example 3.3.3. If $\overline{\rho}$ is semisimple then it admits a universally twistable lift. Indeed, choosing the saturated filtration $(\overline{U_i})_{i=0}^r$ displaying $\overline{\rho}$ as block diagonal with representations $\overline{\rho_1}, \overline{\rho_2}, \ldots, \overline{\rho_r}$ on the diagonal, we know from corollary 1.2.7 that each $\overline{\rho_i}$ lifts to $\mathcal{O}_L$ (for $i = 1, 2, \ldots, r$). We can therefore choose any lifts V_i of $\overline{V_i}$ and a family of block diagonal lifts $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$ to $\mathcal{O}_L$.

We have the following result from [20], which gives some indication as to our interest in representations admitting universally twistable lifts with respect to some increasing filtration.

Proposition 3.3.4. *Suppose $\overline{\rho}$ admits a unverisally twistable lift with respect to an increasing filtration $(\overline{U_i})_{i=0}^r$. Let V_i be any lift of $\overline{V_i}$ for $i = 1, 2, \ldots, r$. Then there exist infinitely many r-tuples $(\psi_1, \psi_2, \ldots, \psi_r)$ such that $\overline{\rho}$ has a lift to $\mathcal{O}_L$ of the form*

$$\rho = \begin{pmatrix} V_1 \bigotimes_{\mathcal{O}_L} \psi_1 & * & \cdots & * \\ 0 & V_2 \bigotimes_{\mathcal{O}_L} \psi_2 & \cdots & * \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & V_r \bigotimes_{\mathcal{O}_L} \psi_r \end{pmatrix};$$

in other words, a lift $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$ to $\mathcal{O}_L$, together with an increasing filtration by $\mathcal{O}_L[G_K]$-submodules $(U(\psi_1, \psi_2, \ldots, \psi_r)_i)_{i=0}^r$ which are free $\mathcal{O}_L$-direct summands and lifts of $(\overline{U_i})_{i=0}^r$, such that $U(\psi_1, \psi_2, \ldots, \psi_r)_i / U(\psi_1, \psi_2, \ldots, \psi_r)_{i-1} \cong V_i \bigotimes_{\mathcal{O}_L} \psi_i$ for all i and the reduction $V(\psi_1, \psi_2, \ldots, \psi_r) \bigotimes_{\mathcal{O}_L} k_L \cong \overline{V}$ induces for all i the reduction $U(\psi_1, \psi_2, \ldots, \psi_r)_i \bigotimes_{\mathcal{O}_L} k_L \cong \overline{U_i}$.

Remark 3.3.5. The key point of this proposition is that there is no restriction on the lifts V_i of $\overline{V_i}$ that we choose, unlike in Definition 3.3.1.

Proof. See Theorem 2.1.7 of [20]. □

We now show that, in addition to semisimple representations as discussed above, the class of Fontaine-Laffaille representations also admit universally twistable lifts.

Theorem 3.3.6. *Suppose K is unramified over $\mathbb{Q}_p$, and let $\overline{\rho} : G_K \longrightarrow GL_n(k_L)$ be a representation with underlying space $\overline{V}$ and an increasing saturated filtration $(\overline{U_i})_{i=0}^r$*

by $k_L[G_K]$-submodules. Assume additionally that $\overline{\rho}$ is Fontaine-Laffaille (in the sense of Definition 2.2.9). Then $\overline{\rho}$ admits a universally twistable lift. Moreover, the lifts $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$ to $\mathcal{O}_L$ are Fontaine-Laffaille for every r-tuple of unramified residually trivial characters $(\psi_1, \psi_2, \ldots, \psi_r)$ of G_K.

Remark 3.3.7. This is essentially the statement of Proposition 2.2.1 of [20]. We now give a proof using the methods of this thesis.

Proof. For $i = 1, 2, \ldots, r$, let $\overline{\rho_i}$ be the representation of G_K whose underlying space is $\overline{V_i}$. Then $\overline{\rho_i}$ is irreducible and Fontaine-Laffaille for all i. Put $n_i = \dim(\overline{\rho_i})$. By Theorem 3.1.1, each $\overline{V_i}$ lifts to a crystalline V_i since $\mathcal{D}_i^{\square,cris}$ is smooth of dimension at least $n_i^2 \geq 1$ from the discussion of the previous section.

Now, given any r-tuple of unramified residually trivial characters $(\psi_1, \psi_2, \ldots, \psi_r)$ of G_K, we may inductively lift $\overline{U_i}$ to an $\mathcal{O}_L[G_K]$-submodule $U(\psi_1, \psi_2, \ldots, \psi_r)_i$ containing $U(\psi_1, \psi_2, \ldots, \psi_r)_{i-1}$ which is a free $\mathcal{O}_L$-direct summand depending only on $(\psi_1, \psi_2, \ldots, \psi_i)$ with $U(\psi_1, \psi_2, \ldots, \psi_r)_i / U(\psi_1, \psi_2, \ldots, \psi_r)_{i-1} \cong V_i \bigotimes_{\mathcal{O}_L} \psi_i$, since, in the notation of Theorem 3.1.5, $\mathcal{F}_{\overline{\rho},i} \longrightarrow \mathcal{F}_{\overline{\rho},i-1} \times \mathcal{D}_i^{\square,cris}$ is smooth of relative dimension at least $n_i \sum_{j<i} n_j \geq 1$, again from the discussion of the previous section.

In this way we have constructed the lift $V(\psi_1, \psi_2, \ldots, \psi_r)$ of $\overline{V}$. $\square$

We note in passing that the proof of this theorem gives an explicit understanding of the freedom we have in constructing the universally twistable lift of $\overline{V}$ as above. In particular, we may lift each $\overline{V_i}$ arbitrarily (and the set of allowable lifts is of rank $n^2([K : \mathbb{Q}_p] + 1) - d_{\overline{V_i}}$ over $\mathcal{O}_L$ for $i = 1, 2, \ldots, r$), and given any lifts, the collection of allowable lifts of $\overline{U_i}$ at each stage i is parametrised by a free $\mathcal{O}_L$-module of explicitly calculable (non-zero) rank.

3.4 Generalisations

We can hope to generalise the theorems of the section 3.2 in several ways. Three particular avenues for generalisation are:

1. Relax the unramified condition on the extension K of $\mathbb{Q}_p$.

2. Consider other classes of representations, such as semistable representations.

3. Relax the condition on the Hodge-Tate weights being inside the Fontaine-Laffaille range.

It seems that all of these questions will need techniques outside those discussed in this thesis to be answered fully; however, in this section we will make a few brief remarks about the second question, and also explore the third question in the situation where the departure from the Fontaine-Laffaille range is relatively "mild".

3.4.1 Semistable representations

A natural approach to the second question would be to establish a category of semi-linear algebra data analogous to Fontaine-Laffaille modules which, in a similar way to Theorem 2.2.8, correspond with subcategories of semistable representations (those representations which are B_{st}-admissible[1] in the sense of Definition 2.1.3). There is some hope here as, in the characteristic zero situation, semistable representations (with coefficients) correspond to so-called "weakly admissible (ϕ, N)-modules" (see Theorem A of [11]).

[1]Here B_{st}, the ring of semistable periods, is a $(\mathbb{Q}_p, G_K)$-regular domain in the sense of Definition 2.1.1. For more details on the construction, see [15] or [8].

However, there are issues with simply adding a "monodromy operator" N to the definition of a Fontaine-Laffaille module, since demanding that $N\phi^{i+1} = \phi^i N$ requires that $N(M^{i+1}) \subseteq M^i$ for all i, but the analogous property is not true of all weakly admissible (ϕ, N)-modules. Indeed, even for $n = 2$, there are irreducible representations whose associated deformation rings are not smooth[2]. See [4] and [5] for an alternative approach, which may give a sufficiently concrete correspondence to allow the methods of this thesis to extend to certain subcategories of semistable representations - sadly, the author has not yet had the opportunity to explore this avenue fully.

3.4.2 Hodge-Tate weights outside the Fontaine-Laffaille range

We now discuss two possible approaches to the third question. However, much as for the case of semistable representations, it should be noted that results analogous to those in this thesis are not expected to be true in a great deal of generality.

(ϕ, Γ)-modules, Wach modules, and crystalline representations

A possible approach to this question would be to seek semilinear algebraic data to categorise crystalline representations where no restrictions on the labelled Hodge-Tate weights are imposed. This is done in [34] and [1]; a good summary, particularly of the generalisation needed to allow coefficients, can be found in [9], upon which the entirety of this subsection is based. See also [22], particularly sections 3.2 and 3.3.

Assume K is unramified over $\mathbb{Q}_p$. Put $H_K = \mathrm{Ker}(\chi_p) \cap G_K$ and $\Gamma_K = G_K/H_K$. We recall the ring R from 2.2.1, together with the element $\epsilon \in R$. Let $\pi = [\epsilon] - 1 \in W(R)$;

[2]The author wishes to thank the referees for this observation.

then the Frobenius ϕ sends π to $(1+\pi)^p - 1$, while G_K acts as

$$g(\pi) = (1+\pi)^{\chi_p(g)} - 1$$

for $g \in G_K$. We let $\mathbf{A}_K$ denote the p-adic completion of $\mathcal{O}_K[[\pi]][\frac{1}{\pi}]$, and $\mathbf{B}_K = \mathbf{A}_K[\frac{1}{p}]$, with a natural action of ϕ and Γ_K obtained by letting ϕ act as Frobenius and Γ_K act trivially on $\mathcal{O}_K$.

If L is a finite extension of $\mathbb{Q}_p$ containing the image of all embeddings $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$, we put $\mathbf{A}_{K,L} = \mathbf{A}_K \bigotimes_{\mathbb{Z}_p} \mathcal{O}_L$ (with the actions of ϕ and Γ_K extended $\mathcal{O}_L$-linearly) and $\mathbf{B}_{K,L} = \mathbf{B}_K \bigotimes_{\mathbb{Q}_p} L$ (with the actions of ϕ and Γ_K extended L-linearly). We then make the following definition.

Definition 3.4.1. 1. A (ϕ, Γ)-*module* over $\mathbf{A}_{K,L}$ is a finitely generated $\mathbf{A}_{K,L}$-module M with continuous semilinear commuting actions of a Frobenius endormophism ϕ_M and of Γ_K. M is *étale* if the image of ϕ_M spans M over $\mathbf{A}_{K,L}$. The category of *étale* (ϕ, Γ)-modules over $\mathbf{A}_{K,L}$ is denoted as $M^{\phi,\Gamma,ét}_{\mathbf{A}_{K,L}}$.

2. A (ϕ, Γ)-*module* over $\mathbf{B}_{K,L}$ is a finitely generated $\mathbf{B}_{K,L}$-module M with continuous semilinear commuting actions of a Frobenius endormophism ϕ_M and of Γ_K. M is *étale* if it contains a ϕ_M-stable $\mathbf{A}_{K,L}$-lattice which is étale over $\mathbf{A}_{K,L}$. The category of *étale* (ϕ, Γ)-modules over $\mathbf{B}_{K,L}$ is denoted as $M^{\phi,\Gamma,ét}_{\mathbf{B}_{K,L}}$.

Our main interest in (ϕ, Γ)-modules is the following result.

Proposition 3.4.2. *There is a tensor-equivalence of categories*

$$\mathbf{D} : Rep_{\mathcal{O}_L}(G_K) \longrightarrow M^{\phi,\Gamma,ét}_{\mathbf{A}_{K,L}}$$

which preserves rank in the sense that $V \in Rep_{\mathcal{O}_L}(G_K)$ is free over $\mathcal{O}_L$ of rank d if and only if $\mathbf{D}(V)$ is free over $\mathbf{A}_{K,L}$ of rank d. Moreover, inverting p leads to an equivalence

$$\mathbf{D}[\frac{1}{p}] : Rep_L(G_K) \longrightarrow M^{\phi,\Gamma,\acute{e}t}_{\mathbf{B}_{K,L}}$$

.

Proof. This is essentially Theorem 3.4.3 of [17], augmented to coefficients in $\mathcal{O}_L$ (or L). For more details, see [9], especially corollary 2.13. $\square$

It remains to determine which objects in $M^{\phi,\Gamma,\acute{e}t}_{\mathbf{A}_{K,L}}$ correspond to crystalline representations. To answer this, let us first define subrings $\mathbf{A}^+_K = \mathcal{O}_K[[\pi]] \subseteq \mathbf{A}_K$ and $\mathbf{B}^+_K = \mathbf{A}^+_K[\frac{1}{p}] \subseteq \mathbf{B}_K$. Just as for the rings $\mathbf{A}_{K,L}$ and $\mathbf{B}_{K,L}$ above, we then put $\mathbf{A}^+_{K,L} = \mathbf{A}^+_K \bigotimes_{\mathbb{Z}_p} \mathcal{O}_L \subseteq \mathbf{A}_{K,L}$ and $\mathbf{B}^+_{K,L} = \mathbf{B}^+_K \bigotimes_{\mathbb{Q}_p} L \subseteq \mathbf{B}_{K,L}$; note that these are all stable under the actions of ϕ and Γ_K. We then make the following definition.

Definition 3.4.3. Let $k \in \mathbb{N}$. A *Wach module* over $\mathbf{A}^+_{K,L}$ (respectively $\mathbf{B}^+_{K,L}$) with weights at most k is a free finite rank $\mathbf{A}^+_{K,L}$-module (respectively $\mathbf{B}^+_{K,L}$-module) N with an action of Γ_K that is trivial modulo π, together with a commuting action ϕ_N of Frobenius on $N[\frac{1}{\pi}]$ such that N is stable by ϕ_N and $N/\phi_N(N)$ is killed by $(\phi(\pi)/\pi)^k$.

We then have the following result.

Proposition 3.4.4. *1. $V \in Rep_L(G_K)$ is crystalline with labelled Hodge-Tate weights between 0 and k if and only if $\mathbf{D}[\frac{1}{p}](V)$ contains a Wach module of rank $dim_L(V)$ with weights at most k. This Wach module is unique if it exists, and is denoted $N(V)$.*

2. For $V \in Rep_L^{cris}(G_K)$, there is a bijection

$$T \mapsto N(V) \cap D(T)$$

between G_K-stable $\mathcal{O}_L$-lattices $T \subseteq V$ and $\mathbf{A}_{K,L}^+$-lattices in $N(V)$ which are Wach modules over $\mathbf{A}_{K,L}^+$.

Proof. See [1], Theorem 2 and [9], especially Corollary 2.19. □

The above result gives some hope that Definition 2.1.28 may be used to define and study crystalline representations with Hodge-Tate weights outside the Fontaine-Laffaille range by understanding Wach modules with A-structure for $A \in \mathcal{C}_L$ (though, as already remarked, one does not expect the results in this thesis to extend completely to his setting). Unfortunately, the author has not yet had the opportunity to explore this avenue fully.

An outlook on smooth representability of crystalline framed deformation functors outside the Fontaine-Laffaille range

Provided that the departure from the Fontaine-Laffaille range is relatively elementary, there are some more concrete results we can give. To approach this, we first establish a more general result on smooth representability of framed deformation functors in the situation where there is no restriction on the extension classes we allow. We then apply this to establish a generalisation of the main results of this chapter in the situation where the Hodge-Tate weights may fall outside the Fontaine-Laffaille range, but where we nevertheless have an understanding of crystalline extensions by mimicking the "Hodge-Tate" case.

Fix representations $\overline{\rho_i} : G_K \longrightarrow GL_{n_i}(k_L)$ $(i = 1, 2)$ and assume that both pairs $(\overline{\rho_1}, \overline{\rho_2})$ and $(\overline{\rho_1}, \chi_p \otimes \overline{\rho_2})$ have no Jordan-Hölder factors in common. Fix any deformation problems $\mathcal{D}^{\square, X_i}_{\overline{\rho_i}}$ $(i = 1, 2)$, together with an extension $\overline{\rho} \in \mathrm{Ext}^1(\overline{\rho_2}, \overline{\rho_1})$, and define a functor $\mathcal{F}^X_{\overline{\rho}} : \mathcal{C}_L \longrightarrow \mathrm{Set}$ sending A to the set of lifts ρ of $\overline{\rho}$ to A with the property that

$$\rho = \begin{pmatrix} \rho_1 & * \\ 0 & \rho_2 \end{pmatrix}$$

for $\rho_i \in \mathcal{D}^{\square, X_i}_{\overline{\rho_i}}(A)$ $(i = 1, 2)$. We then have the following result.

Proposition 3.4.5. *The natural map $\mathcal{F}^X_{\overline{\rho}} \longrightarrow \mathcal{D}^{\square, X_1}_{\overline{\rho_1}} \times \mathcal{D}^{\square, X_2}_{\overline{\rho_2}}$ is smooth of relative dimension $n_1 n_2(1 + [K : \mathbb{Q}_p])$.*

Proof. For $A \in \mathcal{C}_L$ and $\rho_i \in \mathcal{D}^{\square, X_i}_{\overline{\rho_i}}(A)$ $(i = 1, 2)$, put $M_A = \mathrm{Hom}_A(\rho_2, \rho_1)$. As in the proof of Theorem 3.1.5, it suffices to prove that $H^1(G_K, M_A)$ is free over A of rank $d = \dim_{k_L}(H^1(G_K, M_{k_L}))$, and that $d = n_1 n_2 [K : \mathbb{Q}_p]$ (since $\mathrm{Hom}_{G_K}(\rho_2, \rho_1) = 0$ by assumption).

Supposing $x_1, x_2, \ldots, x_r$ generate the maximal ideal $\mathfrak{m}_A$, we get (for an appropriate A-module B) an exact sequence

$$0 \longrightarrow B \longrightarrow A^r \longrightarrow \mathfrak{m}_A \longrightarrow 0$$

of A-modules, where the second map is $(a_1, a_2, \ldots, a_r) \mapsto \sum_j x_j a_j$. Since A is artinian, B is finitely generated as an A-module, and so there is a surjection $A^e \twoheadrightarrow B$ of A-modules, for appropriate integer e. Putting $N_A = M_A \bigotimes_A B$, we get an exact sequence

$$0 \longrightarrow N_A \longrightarrow M_A^r \longrightarrow \mathfrak{m}_A M_A \longrightarrow 0$$

of $A[G_K]$-modules, as well as a G_K-equivariant surjection $M_A^e \twoheadrightarrow N_A$, since M_A is free, and thus flat, over A. Since G_K has cohomological dimension 2, the H^2 functor is right exact and so we see that $H^2(G_K, N_A) = 0$ (using $H^2(G_K, M_A)^e = 0$, which follows from considering the Tate dual of M_A and recalling that $\overline{\rho_1}$ and $\chi_p \otimes \overline{\rho_2}$ have no Jordan-Hölder factors in common).

Having established that $H^2(G_K, N_A) = 0$ it follows by taking cohomology of the exact sequence above that $H^1(G_K, \mathfrak{m}_A M_A) = \mathfrak{m}_A H^1(G_K, M_A)$. Observe further that $\mathfrak{m}_A M_A$ has Tate dual $\mathrm{Hom}_A(\mathfrak{m}_A \rho_1, \chi_p \otimes \rho_2)$, which by assumption has trivial H^0, and so $H^2(G_K, \mathfrak{m}_A M_A) = 0$ by local Tate duality. We conclude by taking cohomology of the exact sequence

$$0 \longrightarrow \mathfrak{m}_A M_A \longrightarrow M_A \longrightarrow M_{k_L} \longrightarrow 0$$

that $H^1(G_K, M_{k_L}) = H^1(G_K, M_A)/\mathfrak{m}_A$.

By Nakayama's lemma, we conclude that $H^1(G_K, M_A)$ is generated by d elements. On the other hand, from the local Euler characterstic formula together with the fact that $H^0(G_K, M_A)$ and $H^2(G_K, M_A)$ both vanish (by local Tate duality and the assumptions on ρ_1 and ρ_2), we see that $H^1(G_K, M_A)$ has size

$$\#H^1(G_K, M_A) = \#\left(\frac{\mathcal{O}_K}{\#M_A}\right) = (\#A)^{n_1 n_2 [K:\mathbb{Q}_p]}$$

from which we conclude that $d = n_1 n_2 [K : \mathbb{Q}_p]$ and that $H^1(G_K, M_A)$ is free over A of rank d, as required. $\square$

We now aim to apply the above result to certain kinds of crystalline representations. We first make the following definition.

Definition 3.4.6. Let a be any integer. A representation $\rho : G_K \longrightarrow GL_n(A)$ is *crystalline with Hodge-Tate weights in the range* $[a, a+p-2]$ if $\rho \bigotimes \chi_p^{-a} \in \mathrm{Rep}_A^{cris,\leq p-2}(G_K)$, where χ_p denotes the cyclotomic character. The *labelled Hodge-Tate weights* of ρ are defined to be those of $\rho \bigotimes \chi_p^{-a}$ with each weight increased by a (and counted with multiplicity). With some abuse of notation we denote the multiset of labelled Hodge-Tate weights for a given label $\sigma : K \hookrightarrow \overline{\mathbb{Q}_p}$ as either $HT_\sigma(\rho)$ or as $HT_\sigma(U_S^{-1}(\rho))$ where convenient.

The situation where the difference between the largest and smallest Hodge-Tate weights exceeds $p-2$ falls outside of the scope of Fontaine-Laffaille theory. However, we can at least make the following definition, inspired by Proposition 2.2.17.

Definition 3.4.7. Let $A \in \mathcal{C}_L$, and suppose $\rho_i : G_K \longrightarrow GL_{n_i}(A)$ $(i = 1, 2)$ are crystalline representations with Hodge-Tate weights in the range $[a_i, a_i + p - 2]$ in the sense of the preceding definition. Suppose further that $\overline{\rho_1} \not\cong \chi_p \bigotimes \overline{\rho_2}$ and that for all σ, $HT_\sigma(\rho_1) > HT_\sigma(\rho_2)$. Then we put $\mathrm{Ext}^1_{cris,A}(\rho_2, \rho_1) = \mathrm{Ext}^1_A(\rho_2, \rho_1)$.

We then have the following result.

Theorem 3.4.8. *For $i = 1, 2, \ldots, r$, let $\overline{\rho_i} : G_K \longrightarrow GL_{n_i}(k_L)$ be irreducible crystalline representations with Hodge-Tate weights in the range $[a_i, a_i + p - 2]$. Suppose further that for all pairs of integers $j < k$, $\overline{\rho_j} \not\cong \chi_p \bigotimes \overline{\rho_k}$ and that for all σ, $HT_\sigma(\rho_j) > HT_\sigma(\rho_k)$. Let $\overline{M_i}$ be the rank n_i Fontaine-Laffaille module associated with $\overline{\rho_i} \bigotimes \chi_p^{-a_i}$. Fix a representation $\overline{\rho}$ which is block upper triangular with $\overline{\rho_1}, \overline{\rho_2}, \ldots, \overline{\rho_r}$*

on the diagonal. Then the functor $\mathcal{F}_{\overline{\rho}}$ as defined in Theorem 3.1.5 is represented by a power series ring over $\mathcal{O}_L$ in $([K:\mathbb{Q}_p]+1)(\sum_{i,j:i\leq j} n_i n_j) - \sum_{i=1}^{r} d_{\overline{M_i}}$ variables.

Proof. This follows from Proposition 3.4.5 and Theorem 3.1.1 analogously to the reasoning in the proof of Corollary 3.1.6, since $HT_\sigma(\rho_j) > HT_\sigma(\rho_k)$ implies that $\overline{\rho_j} \not\cong \overline{\rho_k}$. □

Remark 3.4.9. Note that since every extension is crystalline and we are in the Hodge-Tate case, we should expect an analog of the statement that $d_{\overline{M_{\leq i}},\overline{M_i}} = d_{\overline{M_i}}$ for all i in the above Theorem, so this result is unsurprising.

Bibliography

[1] Laurent Berger, *Limites de représentations cristallines*, Compositio Mathematica **140** (2004), no. 06, 1473–1498.

[2] ______, *Mod p Galois representations, (ϕ,Γ)-modules and representations of $B_2(\mathbb{Q}_p)$*, 2009.

[3] Gebhard Böckle, *Deformations of Galois representations*, Elliptic Curves, Hilbert Modular Forms and Galois Deformations, Springer, 2013, pp. 21–115.

[4] Christophe Breuil, *Construction de représentations p-adiques semi-stables*, Annales scientifiques de l'École Normale Supérieure, vol. 31, 1998, pp. 281–327.

[5] ______, *Représentations semi-stables et modules fortement divisibles*, Inventiones mathematicae **136** (1999), no. 1, 89–122.

[6] Christophe Breuil and William Messing, *Torsion étale and crystalline cohomologies*, Astérisque **279** (2002), 81–124.

[7] Christophe Breuil, Ariane Mézard, et al., *Multiplicités modulaires et représentations de $GL_2(Z_p)$ et de $Gal(\overline{\mathbb{Q}_p}/\mathbb{Q}_p)$ en $l = p$. Appendice par Guy Henniart. Sur l'unicité des types pour GL_2*, Duke Mathematical Journal **115** (2002), no. 2, 205–310.

[8] O. Brinon and B. Conrad, *CMI summer school notes on p-adic Hodge theory (preliminary version)*, June 2009.

[9] Seunghwan Chang and Fred Diamond, *Extensions of rank one (φ, Γ)-modules and crystalline representations*, Compositio Mathematica **147** (2011), no. 02, 375–427.

[10] Laurent Clozel, Michael Harris, and Richard Taylor, *Automorphy for some l-adic lifts of automorphic mod l Galois representations*, Publications mathématiques **108** (2008), no. 1, 1–181.

[11] Pierre Colmez and Jean-Marc Fontaine, *Construction des représentations p-adiques semi-stables*, Inventiones mathematicae **140** (2000), no. 1, 1–43.

[12] Brian Conrad, *Lifting global representations with local properties*, preprint (2011).

[13] Pierre Deligne, *Formes modulaires et representations l-adiques*, Séminaire Bourbaki vol. 1968/69 Exposés 347-363, Springer, 1971, pp. 139–172.

[14] Fred Diamond, Matthias Flach, and Li Guo, *The Tamagawa number conjecture of adjoint motives of modular forms*, Annales scientifiques de l'École Normale Supérieure **37** (2004), no. 5, 663–727 (eng).

[15] Matthew Emerton and Mark Kisin, *Extensions of crystalline representations*, Preprint (1999).

[16] G. Faltings, *Crystalline cohomology and p-adic Galois representations*, Algebraic analysis, geometry, and number theory (Baltimore, MD, 1988), Johns Hopkins Univ. Press, Baltimore, MD, 1989, pp. 25–80.

[17] Jean-Marc Fontaine, *Représentations p-adiques des corps locaux (1ère partie)*, The Grothendieck Festschrift, Springer, 1990, pp. 249–309.

[18] ———, *Le corps de periodes p-adiques*, Périodes p-adiques, vol. 223, Société Mathématique de France, 1994, pp. 59–113.

[19] Jean-Marc Fontaine and Guy Laffaille, *Construction de représentations p-adiques*, Annales scientifiques de l'École Normale Supérieure **15** (1982), no. 4, 547–608 (fre).

[20] Toby Gee, Florian Herzig, Tong Liu, and David Savitt, *Potentially crystalline lifts of certain prescribed types*, arXiv:1506.01050v1 [math.NT] (2015).

[21] Eknath Ghate and Ariane Mézard, *Filtered modules with coefficients*, Transactions of the American Mathematical Society **361** (2009), no. 5, 2243–2261.

[22] S. Hattori, *Ramification of crystalline representations*, Preprint based on talks at the spring school "Classical and p-adic Hodge theories", September 2014.

[23] Mark Kisin, *Moduli of finite flat group schemes, and modularity*, Annals of Mathematics (2009), 1085–1180.

[24] Yurii Ivanovich Manin, *The theory of commutative formal groups over fields of finite characteristic*, Russian Mathematical Surveys **18** (1963), no. 6, 1–83.

[25] B. Mazur, *Deforming Galois representations*, Galois groups over $\mathbb{Q}$, Proc. Workshop, Berkeley/CA (USA) 1987, Publ., Math. Sci. Res. Inst. 16, 385-437 (1989)., 1989.

[26] ———, *An Introduction to the Deformation Theory of Galois Representations*, Modular Forms and Fermat's Last Theorem (G. Cornell, J.H. Silvermann, and G. Stevens, eds.), Springer-Verlag, New York, 1997.

[27] Alain Muller, *Relevements cristallins de représentations galoisiennes*, Ph.D. thesis, Strasbourg, 2013.

[28] Ravi Ramakrishna, *On a variation of Mazur's deformation functor*, Compositio Mathematica **87** (1993), no. 3, 269–286 (eng).

[29] Kenneth A Ribet, *Galois representations and modular forms*, Bulletin of the American Mathematical Society **32** (1995), no. 4, 375–402.

[30] Michael Schlessinger, *Functors of Artin rings*, Transactions of the American Mathematical Society (1968), 208–222.

[31] Gorō Shimura, *Introduction to the arithmetic theory of automorphic functions*, vol. 1, Princeton University Press, 1971.

[32] G. Stevens, *An Overview of the Proof of Fermat's Last Theorem*, Modular Forms and Fermat's Last Theorem (G. Cornell, J.H. Silvermann, and G. Stevens, eds.), Springer-Verlag, New York, 1997.

[33] Takeshi Tsuji, *p-adic étale cohomology and crystalline cohomology in the semi-stable reduction case*, Inventiones mathematicae **137** (1999), no. 2, 233–411.

[34] Nathalie Wach, *Représentations p-adiques potentiellement cristallines*, Bulletin de la Société Mathématique de France **124** (1996), no. 3, 375–400.

[35] Andrew Wiles, *Modular elliptic curves and Fermat's last theorem*, Annals of Mathematics (1995), 443–551.